Strategic Fragmented Markets*

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Abstract

We study the determinants of asset market fragmentation in a model with strategic investors that disagree about the value of an asset. Investors' choices determine the market structure. Fragmented markets are supported in equilibrium when disagreement between investors is low. In this case, investors take the same side of the market and are willing to trade in smaller markets with a higher price impact to face less competition when trading against a dealer. The maximum degree of market fragmentation increases as investors' priors are more correlated. Dealers can benefit from fragmentation, but investors are always better off in centralized markets.

JEL Classification: G12, D43, D47

Keywords: market fragmentation, disagreement, interdealer trading, price impact, demand schedule equilibrium.

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1 Introduction

The structure of financial markets is crucial in determining their efficiency and price dynamics. A key aspect of the market structure is the degree of fragmentation. Indeed, the markets for many financial assets are fragmented. Over-the-counter (OTC) markets, where trading is typically bilateral, are inherently fragmented. Fragmentation is also prevalent in equity markets with stocks being traded in large exchanges, such as NYSE and Nasdaq, as well as in a variety of dark pools and other electronic platforms. Generally, the more venues in which an asset is traded, the more fragmented the market for that asset is. The increase in asset market fragmentation has sparked the interest of regulators and has put the market structure at the center of recent regulatory discussions and proposals. Still, a fundamental question remains unsettled. What are the determinants of market fragmentation?

To address this question, we develop a model which emphasizes the role of investors in determining how fragmented markets are. In particular, we focus on how investors' strategic choices affect the *market structure* in which they trade. In our model, investors trade because they disagree about the value of an asset. The main insight of our paper is that market fragmentation arises when disagreement among investors is low. When disagreement is low, all investors will trade on the same side of the market against a dealer. In this case, an investor has incentives to trade in a smaller market to benefit from a larger share of the gains from trade with the dealer despite having a higher price impact. Investors' choices to trade in smaller markets give rise to fragmented market structures. By focusing on investors' incentives, our approach brings a novel perspective which complements other theories that seek to explain market fragmentation through the lens of trading services providers, such as dealers, exchanges and other trading platforms.

Our model has three dates and a finite number of strategic investors and dealers. Investors are ex-ante homogeneous, but ex-post disagree about the value of an asset that is in zero-net supply. The degree of disagreement in the market is captured by the correlation between investors' priors about the value of the asset. Dealers are homogeneous and do not value the asset intrinsically. Before investors' priors about the asset value are realized, investors choose a dealer with whom to trade and their choices determine the market structure. After the market structure is decided, trade takes place sequentially. First, each dealer and the investors that chose her trade in a local market. Second, dealers participate in an interdealer market. We model both investors' and dealers' trading strategies as quantity-price schedules. When each agent chooses her trading strategy, she understands the impact of her trade on the price (taking all other agents' strategies as given). Each investor also understands that her choice of dealer affects the market structure. Thus, investors act strategically both when markets form, as well as when they trade.

There are three main assumptions in our model: a) the heterogeneity in priors, b) the timing, and c) the trading protocol. First, a pervasive feature of financial markets is to have traders betting against each other based on having opposing views about the future value of an asset. Both academics and regulators consider these heterogeneous beliefs to be an inherent attribute of financial market participants and frequently track disagreement to gauge the health of the financial system. However, our model is also consistent with other interpretations of heterogeneity between investors based on liquidity needs, the use of the asset as collateral, or risk management constraints.

Second, in our set-up investors choices determine a market structure. This is consistent, for instance, with trade formation on swap execution facilities where customers typically initiate requests for trade as Riggs et al. (2018) document. Moreover, by assuming that trade takes place sequentially, we can study the role of the interdealer market for intermediating trade. In fact, trade for various assets takes place in a similar set-up. For instance, Collin-Dufresne, Junge and Trolle (2018) and Duffie, Scheicher and Vuillemey (2015) find evidence that in the CDS market dealers use interdealer markets to manage inventory risk after trading with clients.

Third, representing agents' strategies as quantity-price schedules allows us to capture common elements of the increasingly diverse set of trading protocols that are used in practice in decentralized markets. For instance, in the swap markets, customers interested in trade receive indicative quotes from dealers. However, the final terms of trade adjust to reflect the quantity that the customer wishes to trade. More importantly, a common characteristic of most decentralized markets is that a relatively small number of dealers intermediate a vast proportion of transactions and that trading outcomes reflect dealers' market power relative to other dealers as well as relative to investors. By allowing agents to trade strategically in quantity-price schedules, our model reflects these features.

We obtain three sets of results. Our first set of results concentrates on understanding when fragmented market structures can arise in equilibrium. In our main theorem, we show that a fragmented market structure is an equilibrium when disagreement among investors is low. When choosing to trade in a larger market, investors benefit from a lower price impact but potentially have lower gains from trade with the dealer. The gains from trading with a dealer in a larger market depend on the correlation between investors' priors. When disagreement is low, investors take similar positions against the dealer. This increases the competition among investors, which allows the dealer to exploit her position in the market better. In consequence, the investors' gains from trading with the dealer decrease. The decrease in gains from trade with a dealer in a larger market dominates any improvement in the price impact when investor priors are sufficiently correlated. In this case, market fragmentation is sustained in equilibrium.

Our second set of results explores the role of the dealers' strategic behavior in the interdealer market in attaining market fragmentation by studying two limiting cases of our baseline model: the case of no interdealer market and the case in which the interdealer market is perfectly competitive. To begin with, we show that fragmentation arises in equilibrium even in the absence of an interdealer market. In addition, we show that fragmentation unravels as the interdealer market becomes perfectly competitive. These polar cases show that the dealers' strategic trading behavior is necessary for market fragmentation to arise. Moreover, we examine the maximum degree of fragmentation that can be supported in equilibrium, and find that it is decreasing in the disagreement among investors.

Our third set of results focuses on the liquidity and welfare properties of fragmented and centralized market structures. As is standard, we consider that a market is centralized when all investors and dealers trade in the same market. Our results are consistent with the intuition that assets that are traded in fragmented markets have intrinsically low liquidity, as proxied a by a high correlation between investors' priors. However, a fragmented market structure itself further contributes to lowering the traded volume. Indeed, consistent with empirical findings, trading volume is lower in fragmented markets than in centralized ones keeping the degree of disagreement among investors constant.

Moreover, we analyze investor and dealer welfare when they trade in a fragmented market and compare it to the welfare they would attain if they were to trade in a centralized market. The welfare implications of market fragmentation are distinct for dealers and investors. We show that although dealers benefit from trading in a fragmented market provided investors disagreement is high enough, investors are always better off trading in a centralized market. Thus, trading in a fragmented market can be inefficient.

Lastly, we investigate the predictions of our model using data from equity markets. In particular, we use the dispersion of analyst forecasts about stock prices to proxy for disagreement among investors. Using the fraction of a stock's shares that are traded in dark pools and other alternative trading systems as a measure of market fragmentation .we find that, consistent with our model, stocks for which there is less disagreement trade in more fragmented markets.

Literature Review

This paper relates to several strands of literature. The more relevant studies are those on endogenous market structure and intermediation in decentralized markets.

A series of papers have developed models where the market structure in which assets are traded is endogenously determined. Most of these works emphasize the role of trading services providers, focusing on transaction costs or fees charged by exchanges, or on the competition between venues as potential explanations for market fragmentation. An early contribution is Pagano (1989) who studies a set-up in which traders can choose to enter one of two exchanges in which the same asset is traded. In Rust and Hall (2003) buyers and sellers choose between trading with a market maker at publicly observable bid and ask prices, or with middlemen at privately observed quote prices. In both models, traders concentrate in one market in the absence of fees or transaction costs.[1] In contrast, in our model markets fragmentation arises in equilibrium even when there are no fees or exogenous trading costs.

A related set of papers studies competition between venues. In Pagnotta and Philippon (2018) when two venues compete in the speed with which traders can find counterparties, markets fragmentation arises. Competition also plays a role in segmenting markets in Lester, Rocheteau and Weill (2015), as dealers compete to attract order flow by posting the terms at which they execute trades.[2] Baldauf and Mollner (2019), Cespa and Vives (2019), and Chao, Yao and Ye (2019) propose models in which exchanges compete in the fees charged for trading services and evaluate the impact of fragmentation on market quality. We abstract from competition between venues. Instead, in our model investors choose a dealer with whom to trade based on the size of her local market, which is, in turn, determined by the other investors' choices.

The market structure has also been linked to informational asymmetries. For instance, in Zhu (2014)

[1] In Rust and Hall (2003) consumers and producers are indifferent between trading against a single middleman (concentration), or against a middleman and the market maker (fragmentation) at the Walrasian price.

[2] Although not directly concerned with studying market fragmentation, some other papers analyze models of competition between venues. These studies include Biais (1993), Glosten (1994), Hendershott and Mendelson (2000), Parlour and Seppi (2003), and Santos and Scheinkman (2001).

exchanges attract informed traders who want a fast execution of their order, while uninformed traders, who only have idiosyncratic liquidity needs, trade in dark pools. In contrast, in Kawakami (2017) trading in multiple venues is optimal to avoid excessive information aggregation as revealed risk cannot be traded away. Most recently, Lee and Wang (2018) propose a model in which informed investors trade in exchanges and uninformed hedgers select themselves to trade in OTC markets as dealers are able to attract them with targeted quotes. In our model, there are no information asymmetries, and market fragmentation is driven by the investors' disagreement about the asset value.

Recently, Dugast, Weill and Uslu (2019) explore how heterogeneity in investors' types affects the market structure in which trade occurs. Their model trades off risk sharing and earning intermediation profits. When the latter force dominates, trade takes place in decentralized, over-the-counter markets. In our paper, investors choosing to trade in a fragmented market trade off lower competition for the liquidity in the market, as proxied by the gains from trade with a dealer, and a higher price of that liquidity, as proxied by their price impact. When the former force dominates, trade takes place in fragmented markets.

Some recent papers explore the efficiency of trade in different market structures. In Malamud and Rostek (2017) agents, who take into account their price impact, may benefit from trading in interconnected venues relative to a centralized market. Duffie and Wang (2016) show that OTC markets can be efficient if agents write contingent bilateral contracts. Glode and Opp (2018) illustrate that a market in which agents face costly trading delays can be more efficient than a centralized market in which trade occurs without delays. In contrast to our paper, these models take the market structure as given while we focus on endogeneizing the market structure.

There is a growing literature that studies the role of intermediaries in decentralized markets. Hugonnier, Lester and Weill (2015),Neklyudov (2014) and Chang and Zhang (2016) propose models in which intermediaries facilitate trade between counterparties that otherwise would need to wait a long time to trade. Other papers explore the informational role of intermediaries. In Glode and Opp (2016) the role of intermediaries is to restore efficient trading by reducing adverse selection, while in Boyarchenko, Lucca and Veldkamp (2016) interdealer information sharing improves risk sharing and welfare. Our model complements these works by highlighting the intermediaries' strategic trading behavior as a key determinant of market fragmentation.

The rest of the paper is organized as follows. We present the model in Section 2. In Section 3, we define and characterize the equilibrium of the model when markets are fragmented. In Section 4 we investigate the role of interdealer trading in determining the degree of market fragmentation. We analyze the welfare and liquidity properties in fragmented markets relative to centralized markets in Section 5. We investigate the predictions of our model using data from equity markets in Section 6. Finally, we conclude in Section 7. All omitted proofs are in the Appendix.

2 The model

There are three dates, $t = 0, 1, 2$, and a finite number of agents that trade a risky asset in zero net supply. There are two types of agents, dealers and investors. There are $n_D \geq 3$ dealers indexed by $\ell = 1, ..., n_D$.

We denote by N_D the set of dealers. The utility of a dealer who holds x units of the asset at time date 2 is given by

$$U_D(x) = -\frac{\gamma}{2}x^2.$$

Dealers have no intrinsic value for the asset and are homogenous.

There are also $n_I = n_S \cdot n_D$ investors indexed by $i = 1, ..., n_I$, where n_S is an integer and $n_S \geq 3$. The set of investors is denoted by N_I. An investor i derives utility

$$U_I(x) = \bar{\theta}x - \frac{\gamma}{2}x^2$$

from holding x units of the asset at time 2. $\bar{\theta}$ represents the value of the asset for an investor, which is random and is realized at date 2. The quadratic term in the utility functions for dealers and investors can be interpreted as the cost of holding the asset.

Investors disagree about the value of the asset. At date 1, investors have heterogeneous priors about $\bar{\theta}$ given by

$$\bar{\theta} \sim_i N\left(\theta^i, \sigma_\eta^2\right),$$

where

$$\theta^i = \theta + \eta^i,$$

with $\theta \sim N(0, \sigma_\theta^2)$, and $\eta^i \overset{iid}{\sim} N(0, \sigma_\eta^2)$. Investors' priors over the value of the asset have a common component θ, which can be interpreted as the market sentiment for the asset, and an idiosyncratic component η^i that governs the heterogeneity in beliefs. Both components of the investors' valuations are realized at date 1. The market sentiment θ is observed by all agents when it is realized, while the idiosyncratic component η^i is private information of investor i. Investors take their priors as given and do not learn from the price since they observe the market sentiment θ.[3] We adopt this information structure to abstract from considerations related to learning from prices and focus on the agents' strategic trading behavior as a driver of market fragmentation.

The degree of disagreement among investors is given by the dispersion in the idiosyncratic component of priors, σ_η^2. When $\sigma_\eta^2 = 0$, there is no disagreement and investors have common priors. When $\sigma_\eta^2 \to \infty$ disagreement is maximal. It is helpful to map the degree of disagreement among investors to the correlation in their priors. In the remainder of the paper we will measure the extent to which investors disagree about the value of the asset by $1 - \rho$, where

$$\rho \equiv Corr\left(\theta^i, \theta^j\right) = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\eta^2} \quad \forall i, j \in N_l, i \neq j.$$

When $\sigma_\eta^2 = 0$, the heterogeneity among investors vanishes and $\rho = 1$. When disagreement is maximal $\sigma_\eta^2 \to \infty$, which implies $\rho = 0$.[4]

[3] Heterogeneous beliefs are featured in a large literature following De Long et al. (1990) and Scheinkman and Xiong (2003). Our formulation using heterogeneous priors is closest to Dávila and Parlatore (2019).

[4] The heterogeneity in investors' priors leads to heterogeneous perceived valuations which can also be attributed to differences in liquidity needs, in the use of the asset as collateral, in risk management constraints.

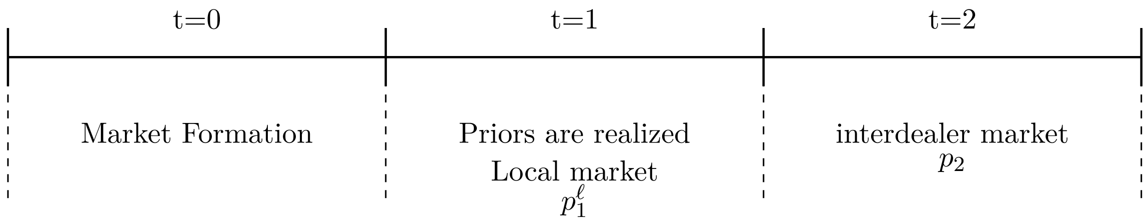


Figure 1: **Timing**

Figure 1 illustrates the timing of the model. Before any heterogeneity among investors is realized, the structure of the market is determined by the investors' choices. At date 0, each investor chooses a dealer with whom to trade. An investor can choose at most one dealer. However, multiple investors can choose the same dealer.[5] Once investors make their dealer selection, markets open and trade takes place in two rounds. At date 1, each dealer ℓ trades with the investors that chose her at date 0, in a local market ℓ. At date 2, dealers trade in an interdealer market. There is a finite number of participants in the local markets and in the interdealer market. Hence, all market participants act strategically and take into account their price impact when making their trading decisions.

The investors' choices at date 0 determine a market structure, m, in which each dealer ℓ interacts with $n^\ell \geq 2$ investors. When an investor i chooses a dealer ℓ, we say that $i\ell \in m$. We denote by $N_I(\ell)$ the set of investors that choose dealer ℓ. A fragmented market structure exists when there are at least three local markets each with $n^\ell \geq 2$ investors. A fragmented market is symmetric when in each local market ℓ there is the same number of investors n_S. We denote by m_{n_S} a symmetric fragmented market. Figure 2 illustrates a symmetric fragmented market structure.

Dealers and investors submit quantity-price schedules when trading, as in Kyle (1989) and Vives (2011). More precisely, the strategy of an agent is a mapping from her information set to the space of demand functions, as follows. The demand function of an investor i with a prior θ^i is a continuous function $X_1^i : \mathbb{R} \to \mathbb{R}$, which maps each price p_1^ℓ in the local market ℓ, into a quantity x_1^i she wishes to trade

$$X_1^i\left(p_1^\ell; \theta^i\right) = x_1^i. \tag{1}$$

A dealer's trading strategy has two components, each corresponding to the markets in which she trades. When trading in a local market ℓ at date 1, the demand function of a dealer ℓ who observes θ, is a continuous function $Q_1^\ell : \mathbb{R} \to \mathbb{R}$ which maps each price p_1^ℓ, into a quantity q_1^ℓ she wishes to trade

$$Q_1^\ell\left(p_1^\ell; \theta\right) = q_1^\ell. \tag{2}$$

At date 2, the demand function of a dealer ℓ who observed θ , is a continuous function $Q_2^\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which maps the quantity, q_1^ℓ, she acquired in the local market and the possible price in the interdealer market, p_2, into a quantity q_2^ℓ she wishes to trade

$$Q_2^\ell\left(p_2; \theta, q_1^\ell\right) = q_2^\ell. \tag{3}$$

[5] Trade in many markets relies on relationships that are very concentrated. Hendershott et al. (2017) document that an investor in the corporate bond market trades on average with six dealers over the course of more than a decade.

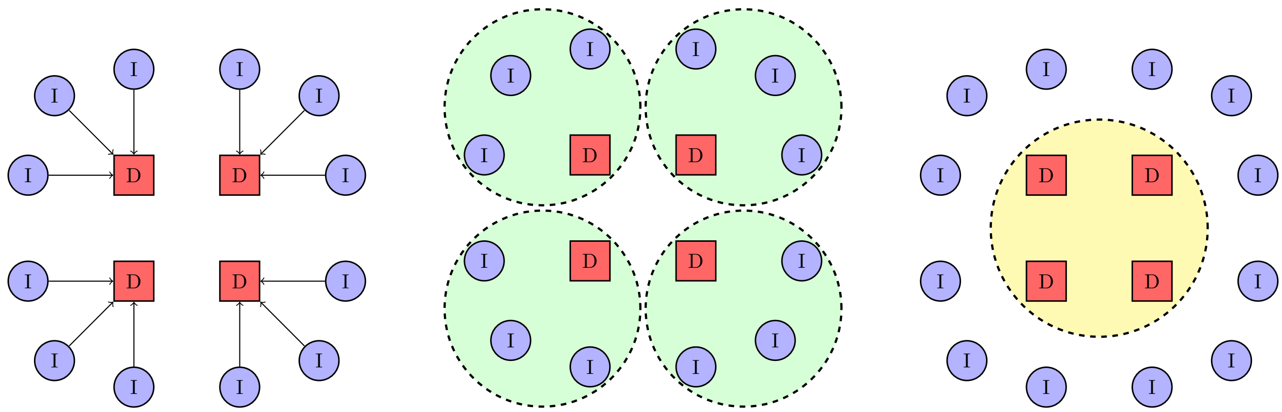


Figure 2: **Fragmented market structure**

Finally, given a market structure m, the expected payoff for an investor i at date 0, corresponding to the strategy profile $\left\{X_1^i, Q_1^\ell, Q_2^\ell\right\}_{i\in N_I,\ell\in N_D}$ is

$$V_1^i\left(m\right) = \mathbb{E}_0\left[\theta^i X_1^i\left(p_1^\ell;\theta^i\right) - \frac{\gamma}{2}\left(X_1^i\left(p_1^\ell;\theta^i\right)\right)^2 - X_1^i\left(p_1^\ell;\theta^i\right)p_1^\ell\right], \tag{4}$$

where p_1^ℓ is the price at which local market ℓ clears, i.e., p_1^ℓ is such that

$$Q_1^\ell\left(p_1^\ell;\theta\right) + \sum_{i\in N_I(\ell)} X_1^i\left(p_1^\ell;\theta^i\right) = 0, \quad \ell\in N_D. \tag{5}$$

Similarly, the expected payoff of a dealer ℓ at date 0 is

$$V_1^\ell\left(m\right) = \mathbb{E}_0\left[-\frac{\gamma}{2}\left(Q_1^\ell\left(p_1^\ell;\theta\right) + Q_2^\ell\left(p_2;\theta,q_1^\ell\right)\right)^2 - Q_2^\ell\left(p_2;\theta,q_1^\ell\right)\cdot p_2 - Q_1^\ell\left(p_1^\ell;\theta\right)\cdot p_1^\ell\right], \tag{6}$$

where p_2 is the price at which the interdealer market clears

$$\sum_{\ell\in N_D} Q_2^\ell\left(p_2;\theta,q_1^\ell\right) = 0. \tag{7}$$

3 Equilibrium

In this section, we define and characterize the equilibrium. We start by computing the equilibrium in the interdealer market at date 2 taking as given a market structure m and the dealers' choices in their local markets at date 1. Then, we characterize the equilibrium in the local market given a market structure m. Lastly, we look at the equilibrium conditions in the market formation game which determines the equilibrium market structure, m. All omitted proofs are in the Appendix.

Definition 1. (Equilibrium) An equilibrium is a market structure m and demand functions $\left\{Q_1^\ell, Q_2^\ell\right\}_{\ell\in N_D}$ for dealers and $\{X_1^i\}_{i\in N_I}$ for investors such that given the pricing functions in Eq. (5) and Eq. (7)

1. Q_2^ℓ solves each dealer ℓ's problem in the interdealer market at date 2

$$\max_{Q_2^\ell} \left[-\frac{\gamma}{2} \left(q_1^\ell + Q_2^\ell \right)^2 - p_2 Q_2^\ell \right] \tag{8}$$

taking as given the other dealers' demand functions in the interdealer market $\left\{ Q_2^l \right\}_{l \in N_D, l \neq \ell}$;

2. Q_1^ℓ solves each dealer ℓ's problem in her local market at date 1

$$\max_{Q_1^\ell} \mathbb{E}_1 \left[-\frac{\gamma}{2} \left(Q_1^\ell + Q_2^\ell \left(p_2; \theta, Q_1^\ell \right) \right)^2 - p_2 Q_2^\ell \left(p_2; \theta, Q_1^\ell \right) \right] \tag{9}$$

taking as given investors' demand function in local market ℓ, $\{X_1^i\}_{i \in N_I(\ell)}$;

3. X_1^i solves each investor i's problem in the local market at date 1

$$\max_{X_1^i} \left[\theta^i X_1^i - \frac{\gamma}{2} \left(X_1^i \right)^2 - p_1^\ell X_1^i \right] \tag{10}$$

taking as given dealer ℓ's and the other investors' demand functions in local market ℓ, Q_1^ℓ and $\left\{ X_1^j \right\}_{j \in N_I(\ell), j \neq i}$, respectively; and

4. no investor i in local market ℓ benefits from deviating and joining a different local market $\ell' \neq \ell$, for all $\ell, \ell' \in N_D$, i.e., the expected payoff an investor receives from deviating to the market structure $(m - i\ell + i\ell')$ is not larger than the expected payoff she obtains in the market structure m, for any $\ell' \neq \ell$,

$$V_1^i \left(m - i\ell + i\ell' \right) \leq V_1^i \left(m \right) \quad \forall i \in N_I \left(\ell \right), \forall \ell \in N_D. \tag{11}$$

Our notion for the equilibrium market structure, described in condition 4 in Definition 1, is related to the concept of pairwise stability introduced in Jackson and Wolinsky (1996). Since there is a finite number of agents, all agents trade strategically and take into account their price impact when submitting their demand. For the same reason, when an investor evaluates the benefit of leaving the local market ℓ and joining the local market ℓ', she understands that trading outcomes in the market structure $m - i\ell + i\ell'$ are different than in the market structure m. To keep our analysis tractable, we restrict our attention to equilibria in which the market structure is symmetric and agents have linear trading strategies.

3.1 Interdealer market

At date 2, after each dealer trades with the investors that chose her, the interdealer market opens. A dealer ℓ enters the interdealer market with an inventory q_1^ℓ of the asset, which she acquired in local market ℓ. Each dealer is privately informed about her inventory and knows that the other dealers trade optimally in their local markets. In the interdealer market, dealers choose their trading strategies taking as given the other dealers' demand functions, as well as the distribution of their inventories acquired at date 1. We simplify the optimization problem in Eq. (8), which is defined over a function space, to finding the functions $Q_2^\ell \left(p_2; \theta, q_1^\ell \right)$ point-wise. To do this, we fix the realization of the set of idiosyncratic shocks to priors $\{\eta^i\}_{i \in N_I}$. This maps into a realization of inventories $\left\{ q_1^\ell \right\}_{\ell \in N_D}$ that dealers bring to the interdealer market. Then, we solve for the optimal quantity that dealer ℓ demands in the interdealer market as she takes as given the demand functions of the other dealers. This procedure allows us to derive the optimal demand function of dealer ℓ point by point. We formalize the argument below.

The first order condition for dealer ℓ is

$$\gamma\left(q_1^\ell + q_2^\ell\right) + p_2 + \frac{\partial p_{2,-\ell}}{\partial q_2^\ell} q_2^\ell = 0, \tag{12}$$

where $p_{2,-\ell}$ is the inverse residual demand of dealer ℓ implied by

$$\sum_{l \in N_D, l \neq \ell} Q_2^l\left(p_2; \theta, q_1^l\right) + q_2^\ell = 0.$$

Since holding the asset has no intrinsic value for dealers, each dealer ℓ chooses q_2^ℓ to minimize her cost of holding the asset, net of any cash transfers in the interdealer market. The first two terms on the left-hand side of Eq. (12) represent the direct costs of demanding an additional unit of the asset in the interdealer market. The first term represents the marginal increase in the holding cost whereas the second term is the cost of purchasing an additional unit of the asset. Since the interdealer market is strategic, there is an additional, indirect cost of increasing the quantity demanded: the impact this quantity has on the market clearing price. The third term in the first order condition for dealers captures this indirect cost. The following proposition establishes the existence and uniqueness of the equilibrium in the interdealer market.

Proposition 1. ***(Existence and Uniqueness)*** *Given a market structure m and inventories $\left\{q_1^\ell\right\}_{\ell \in N_D}$, there exists a unique symmetric equilibrium in linear strategies in the interdealer market.*

The equilibrium in the interdealer market is straightforward. The first order condition in Eq. (12) implies that the demand function of a dealer ℓ is

$$Q_2^\ell\left(p_2; \theta, q_1^\ell\right) = -\frac{1}{\gamma + \lambda_2^\ell}\left(\gamma q_1^\ell + p_2\right), \tag{13}$$

where $\lambda_2^\ell \equiv \frac{\partial p_{2,-\ell}}{\partial q_2^\ell}$ is dealer ℓ's price impact in the interdealer market. In equilibrium, dealer ℓ's price impact is given by

$$\lambda_2^\ell = \frac{\gamma}{n_D - 2}. \tag{14}$$

The equilibrium price in the interdealer market is

$$p_2 = -\gamma \frac{\sum_{l \in N_D} q_1^l}{n_D}. \tag{15}$$

From Eq. (13) and Eq. (15) it follows that the equilibrium quantity traded by dealer ℓ when the dealers' inventories are $\left\{q_1^l\right\}_{l \in N_D}$ is

$$q_2^\ell = \frac{\gamma}{\gamma + \lambda_2^\ell}\left(\frac{\sum_{l \in N_D} q_1^l}{n_S} - q_1^\ell\right). \tag{16}$$

A dealer trades in the interdealer market to off-load some of her inventory as she faces quadratic holding costs. If dealers faced linear holding costs there would be no trade in the interdealer market. The term between parenthesis in Eq. (16) captures the gains from trading in the interdealer market for an individual dealer. The larger the difference between the average inventory in the market and an individual dealer's inventory, the larger the amount the individual dealer will trade. If all dealers hold the same amount of inventory at the beginning of date 2, trade in the interdealer market breaks down.

However, even when there is scope for off-loading inventories, a dealer restricts her trade because she has an impact on the price. As the first term in Eq. (16) shows, a larger price impact λ_2^ℓ makes dealers less willing to trade. The price impact captures how strategic the interdealer market is and it depends on the market structure only through the number of dealers participating in the market. The larger the number of dealers in the interdealer market, the less strategic (the deeper) the interdealer market is, and the lower the price impact of each dealer. Therefore, for a given level of heterogeneity in inventories, dealers will trade more in deeper markets. As $n_D \to \infty$, and the market becomes perfectly competitive, all dealers hold the same amount of the asset at the end of period 2 irrespective of their inventory choice at date 1. In this case, dealers can share the idiosyncratic shocks they face in the local markets perfectly.

3.2 Local markets

At date 1, after each investor chooses a dealer with whom to trade and all idiosyncratic priors are realized, strategic local (investor-dealer) markets open. Each market ℓ is comprised of dealer ℓ and the n^ℓ investors who chose to trade with her. Each of these market participants chooses her trading strategy optimally taking the other participants' trading strategies as given. As in the interdealer market, we solve for the demand functions that solve the optimization problems in Eq. (10) for investors and Eq. (9) for dealers point-wise.

Investors. The first order condition for an investor i in local market ℓ is

$$\theta^i - p_1^\ell - \gamma x_1^i - \frac{\partial p_{1,-i}^\ell}{\partial x_1^i} x_1^i = 0, \tag{17}$$

where $p_{1,-i}^\ell$ is the inverse residual demand function for investor i implied by

$$\sum_{j \in N_I(\ell), j \neq i} X_1^j \left(p_{1,-i}^\ell; \theta^j\right) + x_1^i + Q_1^\ell \left(p_{1,-i}^\ell; \theta\right) = 0.$$

Each investor i demands a quantity x_1^i so that her marginal utility equalizes her marginal cost of trading. The first term in Eq. (17) is the marginal benefit of increasing the final asset holdings for an investor i, which is given by her prior θ^i. The following three terms in Eq. (17) represent investor i´s marginal cost of increasing her demand. The second and third terms represent the price the investor pays to acquire an additional unit of the asset and the marginal increase in her holding costs, respectively. The last term is investor i's price impact, which captures the cost of trading in a strategic market.

Dealer. The first order condition for dealer ℓ in the local market is

$$\frac{dV_2^\ell(m)}{dq_1^\ell} - p_1^\ell - \frac{\partial p_{1,-\ell}^\ell}{\partial q_1^\ell} q_1^\ell = 0, \tag{18}$$

where $V_2^\ell(m)$ represents the payoff that dealer ℓ expects to receive in the interdealer market

$$V_2^\ell(m) = \mathbb{E}_1 \left[-\frac{\gamma}{2} \left(Q_2^\ell \left(p_2; \theta, q_1^\ell\right) + q_1^\ell \right)^2 - p_2 Q_2^\ell \left(p_2; \theta, q_1^\ell\right) \middle| \theta, p_1^\ell \right],$$

where p_2 is the equilibrium price and q_2^ℓ is the equilibrium quantity traded by dealer ℓ in the interdealer market.

Since all agents are strategic, each dealer ℓ takes into account the effect of her trade on the inverse residual demand function, $p_{1,-\ell}^\ell$, in her local market which is implied by

$$\sum_{j \in N_I(\ell)} X_1^j \left(p_{1,-\ell}^\ell, \theta^i\right) + q_1^\ell = 0.$$

Analogous to the investor's problem, dealer ℓ will demand a quantity q_1^ℓ to equalize the marginal benefit and the marginal cost associated with increasing the quantity demanded. Dealers do not attach any intrinsic value to the asset. However, since dealers can access both the local and interdealer markets, they can benefit from the price differences across both markets. By participating in both markets, dealers intermediate trade between investors within the local market and across different local markets through the interdealer market. The first term in Eq. (18) is the expected marginal benefit of increasing the quantity that the dealer acquires in the local market and it represents the dealer's *willingness to intermediate*. The second term captures the pecuniary cost of increasing q_1^ℓ by one unit and the third term captures the cost of trading in strategic markets, measured by the dealer's price impact. The next proposition establishes the existence and uniqueness of the equilibrium in the local markets.

Proposition 2. ***(Existence and uniqueness)*** *Given a market structure m, there exists a unique symmetric equilibrium in linear strategies at date* 1*.*

To compute the equilibrium at date 1, we derive the equilibrium trading strategies of the investors and the dealer in each local market ℓ. The first-order condition in Eq. (17) implies that the demand function of an investor i in local market ℓ is

$$X_1^i \left(p_1^\ell; \theta^i\right) = \frac{\theta^i - p_1^\ell}{\gamma + \nu_1^\ell}, \tag{19}$$

where $\nu_1^\ell \equiv \frac{\partial p_{1,-i}^\ell}{\partial x_1^i}$ is investor i's price impact in local market ℓ. Similarly, the first-order condition for dealer ℓ in her local market in Eq. (18) implies that the demand function of the dealer ℓ is

$$Q_1^\ell \left(p_1^\ell; \theta\right) = \frac{1}{\lambda_1^\ell} \left(\frac{dV_2^\ell(m)}{dq_1^\ell} - p_1^\ell\right), \tag{20}$$

where $\lambda_1^\ell \equiv \frac{\partial p_{1,-\ell}^\ell}{\partial q_1}$ represents dealer ℓ's price impact in her local market. The quantity that an agent trades in a local market is proportional to her perceived marginal gain of holding the asset, which is given by the difference between her expected marginal valuation for the asset at date 1 and the price of the asset in the local market. An investor's marginal valuation is simply her prior θ^i, while a dealer's marginal valuation for the asset is given by her expected payoff from bringing an additional unit of the asset to the interdealer market, $\frac{dV_2^\ell}{dq_1^\ell}$. As in the interdealer market, the price impact in the local market restricts the amount traded by dealers and investors. The following lemma characterizes the investors' and dealer's equilibrium price impact in the local market.

Lemma 1. ***(Equilibrium price impacts)*** *In each local market ℓ, the investors' and the dealer's equilibrium price impact satisfy the following system of equations*

$$\nu_1^\ell = \frac{1}{\frac{n^\ell - 1}{\gamma + \nu_1^\ell} + \frac{\gamma + \lambda_2^\ell}{\gamma\left(\lambda_2^\ell + \lambda_1^\ell\right) + \lambda_1^\ell \lambda_2^\ell + \frac{\gamma^2}{n_D}}} \quad \text{and} \quad \lambda_1^\ell = \frac{\gamma + \nu_1^\ell}{n^\ell}. \tag{21}$$

Moreover, the price impacts satisfy the following properties

$$a)\frac{\partial \nu_1^\ell}{\partial n^\ell} < 0, \quad b)\frac{\partial \lambda_1^\ell}{\partial n^\ell} < 0, \quad c)\frac{\partial \nu_1^\ell}{\partial n_D} < 0, \quad \text{and} \quad d)\frac{\partial \lambda_1^\ell}{\partial n_D} < 0.$$

Lemma 1 shows that investors and dealers trade more aggressively when local markets are larger, and when the interdealer market is less strategic (i.e., deeper). As it is usual in models of strategic trading, the larger the number of investors in the local market, the lower the price impact of a market participant and the more investors and dealers react to the price. When the interdealer market is less strategic, it is less costly for the dealer to unload her inventory at date 2, which reduces her cost of holding inventory from date 1 to date 2 making the dealer trade more aggressively. In turn, the dealer's lower sensitivity to the price implies a flatter inverse residual demand that translates into a lower price impact for investors.

Using Eq. (21) in Lemma 1 and Eq. (14), it can be seen that the price impact in local market ℓ for the investors, ν_1^ℓ, and for the dealer, λ_1^ℓ, depend on the market structure m only through the number of dealers participating in the interdealer market, n_D, and the number of investors in the local market ℓ, n^ℓ, but not through the number of investors in other local markets. Nevertheless, the equilibrium outcomes in the local markets depend on the whole market structure.

However, the structure of the market affects investors' and dealers' demands through different mechanisms. The investors' equilibrium demands depend on the market structure m only through an investor's price impact ν_1^ℓ, as can be seen from Eq. (19). The dealer's equilibrium demand in her local market in Eq. (20) depends on the market structure through her price impact λ_1^ℓ, as well as through the dealer's willingness to intermediate, as given by the marginal value of bringing an additional unit to the interdealer market $\frac{dV_2^\ell(m)}{dq_1^\ell}$. The dealer's price impact determines the slope of the dealer's inverse demand while her willingness to intermediate determines its intercept.

The dealer's willingness to intermediate depends on the market structure through her price impact in the local market and through the equilibrium in the interdealer market. Indeed, the dealer's marginal valuation of the asset depends on the gains from trade she expects to attain in the interdealer market, which depends on number of dealers and on the distribution of investors across local markets. Differentiating V_2^ℓ, using Eq. (12) and substituting in dealer ℓ's demand function in the local market in Eq. (20), as well the price in the local market in Eq. (26), we obtain that the dealer's expected marginal valuation for the asset is given by

$$\frac{dV_2^\ell(m)}{dq_1^\ell} = w^\ell \frac{\sum_{i \in N_I(\ell)} \theta^i}{n^\ell} + \left(1 - w^\ell\right) \frac{\gamma}{\gamma + \lambda_2^\ell} \mathbb{E}\left[p_2 \middle| \theta, p_1^\ell\right], \tag{22}$$

where the weight w^ℓ is given by

$$w^\ell = \frac{\gamma \lambda_2^\ell}{\gamma\left(2\lambda_1^\ell + \lambda_2^\ell\right) + 2\lambda_1^\ell \lambda_2^\ell}.$$

As one can see from Eq. (22), dealer ℓ's marginal valuation of the asset has two components. The first component, $\frac{\sum_{i\in N_I(\ell)}\theta^i}{n^\ell}$, represents the benefit from trading with the investors in her local market, while the second component, $\frac{\gamma}{\gamma+\lambda_2^\ell}\mathbb{E}\left[p_2|\,\theta,p_1^\ell\right]$, is the benefit from intermediating by trading in the interdealer market. The weight the dealer attaches to these two components depends on her price impacts in the local and interdealer markets. When the dealer's price impact impact in the local market λ_1^ℓ increases, the dealer weighs the gains from trade in the local market less in her marginal valuation of the asset, i.e., w^ℓ decreases. Analogously, when the dealer's price impact in the interdealer market λ_2^ℓ increases, the dealer weighs the gains from trade in the interdealer market less in her marginal valuation of the asset, i.e., w^ℓ increases. If trading in the interdealer market is perfectly competitive and $\lambda_2^\ell=0$, the dealer only cares about her trades in the interdealer market and $w^\ell=0$.

The expected price at date 2 is determined by the extent of the opportunities to re-trade among dealers in the interdealer market, which depends on the market structure. More specifically, dealer ℓ expects the price in the interdealer market to be

$$\mathbb{E}\left[p_2|\,\theta,q_1^\ell\right]=-\frac{\gamma}{n_D}\left(q_1^\ell+\sum_{l\in N_D,l\neq\ell}\mathbb{E}\left[q_1^l\middle|\,\theta\right]\right). \tag{23}$$

The expected price Eq. (23) depends on dealer ℓ's trade in her local market and on all the other dealers' expected trades in their local markets, which, in turn, depend on the conditions in their local markets. More specifically, the expected amount a dealer trades in her local market depends on the number of investors in the local market and on the degree of heterogeneity in investors' priors, ρ. As we show in Theorem 1, the degree of heterogeneity in investors' priors, ρ, is a key determinant of the equilibrium market structure .

3.3 Market Formation

At date 0, before any uncertainty is realized, each investor i chooses a dealer with whom to trade. Since each investor i takes the other investors' choices as given, from investor i's perspective, choosing a dealer with whom to trade is the same as choosing between two market structures. A symmetric fragmented market structure is an equilibrium if

$$\Delta^i\left(\rho;n_D\right)\equiv V_1^i\left(m_{n_S}\right)-V_1^i\left(m_{n_S}-i\ell+i\ell'\right)>0,\quad \forall i\in N_I,\forall\ell'\neq\ell,\forall\ell,\ell'\in N_D, \tag{24}$$

where $\Delta^i\left(\rho;n_D\right)$ represents the marginal benefit for investor i of participating in symmetric market structure, m_{n_S}, relative to participating in market structure $(m_{n_S}-i\ell+i\ell')$, when the correlation across investor priors is ρ and when there are n_D dealers. Effectively, an investor weighs the benefit of trading in a local market ℓ against one dealer and other n_S-1 investors, against the benefit of trading in a local market ℓ' against one dealer and other n_S investors. In doing so, she also anticipates that dealers trade in the interdealer market.

The expected utility of an investor of participating in a local market with n^ℓ investors when the market structure is m is given by

$$V_1^i\left(m\right)=\left(\frac{\gamma}{2}+\nu_1^\ell\right)\mathbb{E}\left[\left(x_1^i\right)^2\right]=\frac{\frac{\gamma}{2}+\nu_1^\ell}{\left(\gamma+\nu_1^\ell\right)^2}\mathbb{E}\left[\left(\theta^i-p_1^\ell\left(m,\theta\right)\right)^2\right], \tag{25}$$

where ν_1^ℓ and $p_1^\ell(m,\theta)$ are, respectively, the price impact and the equilibrium price in local market ℓ when the aggregate component of the priors is θ and the market structure is m.

Writing the expected utility of investors as in Eq. (25) shows that the market structure m affects the investors' expected utility through the price impact she will face in her local market as well as through the price at which she expects to trade. Substituting the investors' and dealer's demand functions, respectively in Eq. (19) and Eq. (20), into the market clearing condition in Eq. (5), and using the expression for the dealer's price impact in Eq. (21), the price in the local market is

$$p_1^\ell(m,\theta) = \frac{\frac{1}{\gamma+\nu_1^\ell}}{\frac{1}{\gamma+\nu_1^\ell}+\frac{1}{\lambda_1^\ell}} \sum_{j\in N_I(\ell)} \theta^j + \frac{\frac{1}{\lambda_1^\ell}}{\frac{1}{\gamma+\nu_1^\ell}+\frac{1}{\lambda_1^\ell}} \frac{dV_2^\ell}{dq_1^\ell}(m,\theta) = \frac{1}{2}\left(\frac{\sum_{j\in N_I(\ell)}\theta^j}{n^\ell} + \frac{dV_2^\ell}{dq_1^\ell}(m,\theta)\right). \tag{26}$$

Eq. (26) shows that the price is a weighted average of each market participant's marginal valuation, where each weight is given by the relative slopes of an agent's demand function and the aggregate demand. Everything else equal, when the price impact of an agent increases, her demand becomes less responsive to the price. In consequence, the agent's marginal valuation weights less in the equilibrium price. Using Lemma 1 one can see that the dealer's demand is as elastic as the aggregate demand of investors and, hence the equilibrium price in the local market is an average of the investors' average priors, $\frac{\sum_{i\in N_I(\ell)}\theta^i}{n^\ell}$, and the dealer's willingness to intermediate, $\frac{dV_2^\ell}{dq_1^\ell}(m,\theta)$.

Using the expression for the equilibrium price in the local market in Eq. (26) in Eq. (25) we have that the expected utility of an investors of participating in a local market ℓ when the market structure is m can be written as

$$V_1^i(m) = \frac{\frac{\gamma}{2}+\nu_1^\ell}{\left(\gamma+\nu_1^\ell\right)^2}\mathbb{E}\left[\left(\frac{1}{2}\left(\theta^i - \frac{\sum_{j\in N_I(\ell)}\theta^j}{n^\ell}\right) + \frac{1}{2}\left(\theta^i - \frac{dV_2^\ell(m)}{dq_1^\ell}\right)\right)^2\right]. \tag{27}$$

The first term in the investor's expected utility captures the cost of trading in a strategic market. Intuitively, everything else equal, an investor's expected utility is higher when her price impact in the local market, ν_1^ℓ, is lower. The second term in Eq. (27) captures the expected gains from trade in the local market for an investors i. The expected gains from trade have two components. The term $\theta^i - \frac{\sum_{j\in N_I(\ell)}\theta^j}{n^\ell}$ captures the gains that the investor obtains from trading with other investors in her local market, while the term $\theta^i - \frac{dV_2^\ell(m)}{dq_1^\ell}$ captures the gains that the investor obtains from trading with the dealer.

When choosing whether to deviate from a given market structure, an investor takes into account her effect on her price impact, on the gains from trade with the other investors in her local market, and with her dealer. The following theorem states when symmetric market structures arise in equilibrium.

Theorem 1. ***(Market fragmentation)*** *There exists a threshold $\rho^\star \in [0,1)$ and a number of dealers $n_D^\star(\rho^\star, n_I)$ such that for all $\rho > \rho^\star$ the fragmented symmetric market structure with $n_D^\star(\rho^\star, n_I)$ dealers and n_I investors is an equilibrium.*

Theorem 1 shows that if investor priors are sufficiently positively correlated a fragmented market structure is an equilibrium and no individual investor has incentives to deviate to a larger local market. To understand the intuition behind this result, we first describe the trade-off that an investor faces by choosing to trade in a larger market.

In a larger local market, an investor has a lower price impact and higher gains from trade with the other investors, which, everything else equal, increase her expected utility. However, whether the gains of trading with a dealer in a larger market are higher depends on the correlation between investors' priors. Indeed, in a larger market the dealer has a lower price impact, which tilts her marginal valuation for the asset towards the average in the investors' priors, as we have explained in Section 3.2. Everything else equal, this decreases the investor's gains from trading with the dealer. At the same time, in a larger market the average prior of the investors is closer to the common component of the priors which increases the gains from trading with the dealer. The relative strength of these two effects is determined by the correlation in investors' priors ρ. Hence, there may be a trade off between lower price impacts and higher gains from trade with investors, and lower gains from trade with the dealer that determines an investor's incentives to deviate from a market structure m, and as Theorem 1 shows, it depends on the correlation in the investors' priors ρ.

The change in the gains from trading with the dealer in a larger market is easiest to understand in the limiting case in which investors have common priors and $\rho \to 1$. In this case, an additional investor in the local market will trade the exact same amount as the other investors. While this does not affect the average prior in the local market, it decreases the gains from trading with a dealer. This is because a dealer's price impact will be lower in a larger local market which will shift the dealer's marginal valuation closer to θ. Put differently, the competition among investors increases, which allows the dealer to better exploit her position in the market. Hence, even though a dealer's lower price impact increases the amount she trades, the increase is less than proportional to the size of the market.

As ρ departs from 1 and investors disagree about the value of the asset, the loss in gains from trading with a dealer in a larger local market decreases. However, this loss in gains from trade still outweighs the benefit from having a smaller price impact and higher gains from trading with more investors, as long as the correlation in investors' priors is high enough, i.e. $\rho > \rho^{\star}$. In this case, deviating from a fragmented symmetric market structure is not profitable for any investor.

4 Interdealer trading and market fragmentation

While it is investors' choices that determine the market structure, the strategic behavior of dealers also plays a crucial role in determining the equilibrium market structure. In this section we examine the role of the interdealer market for market fragmentation. We start by analyzing two polar cases: the case of a perfectly competitive interdealer market, and the case of no interdealer market. As we explain below, the first case is equivalent to taking the limit of dealer ℓ's price impact in the interdealer market $\lambda_2^{\ell} \to 0$ in our main model, while the second case is equivalent to taking the limit of dealer ℓ's price impact in the interdealer market $\lambda_2^{\ell} \to \infty$.

4.1 Perfectly competitive market

In our framework the interdealer market becomes perfectly competitive when the number of dealers grows large, i.e., when $n_D \to \infty$. The following proposition shows the effect of interdealer trading on the equilibrium market structure in this limit.

Proposition 3. ***(Competitive interdealer market)*** *No fragmented symmetric market structure can be supported in equilibrium as $n_D \to \infty$.*

Proposition 3 shows that dealers strategic trading behavior is a key determinant of market fragmentation. Indeed, as $n_D \to \infty$ and the interdealer market becomes perfectly competitive, the price impact of each dealer in the interdealer market goes to 0. This implies that the marginal cost of unloading inventories in the interdealer market is independent of the size of the inventories. Therefore, it becomes cheaper for dealers to intermediate an additional unit and dealers are willing to trade more in larger local markets. Joining a larger local market becomes desirable for an investor, as she benefits from a lower price impact and higher gains from trade with other investors without facing lower gains from trade with the dealer. Formally, we have that $\lim_{n_D \to \infty} \Delta^i (\rho; n_D) < 0$.

4.2 No interdealer market

The other limiting case is when there is no interdealer market. The set-up is identical to the one described in Section 2, except that dealers do not have any opportunity to re-trade at date $t = 2$. Thus, at $t = 0$ investors choose a dealer with whom to trade, and at $t = 1$ each dealer trades with the investors that chose her in a local market. Even though they do not value the asset intrinsically and do not have the opportunity to intermediate between local markets, dealers are willing to trade with investors provided the price is favorable to them. In particular, each dealer ℓ chooses a demand function, $Q_1^\ell \left(p_1^\ell; \theta\right)$, to maximize her objective function[6]

$$\max_{Q_1^\ell} -\frac{\gamma}{2}\left(Q_1^\ell\left(p_1^\ell;\theta\right)\right)^2 - p_1^\ell Q_1^\ell\left(p_1^\ell;\theta\right),$$

taking into account the effect of her trade on the inverse residual demand function, $p_{1,-\ell}^\ell$, in her local market which is implied by

$$\sum_{j \in N_I(\ell)} X_1^j\left(p_{1,-\ell};\theta,\eta^i\right) + q_1^\ell = 0.$$

The first-order condition for dealer ℓ implies that her optimal demand function is

$$Q_1^\ell\left(p_1^\ell;\theta\right) = -\frac{p_1^\ell}{\gamma+\lambda_1^\ell}, \tag{28}$$

where $\lambda_1^\ell = \frac{\partial p_{1,-\ell}}{\partial q_1^\ell}$ represents her price impact in local market ℓ. The investors' optimization problem is the same as described in Section 3.2. That is, the optimal demand of an investor i in local market ℓ is given by

$$X_1^i\left(p_1^\ell;\theta^i\right) = \frac{\theta^i - p_1^\ell}{\gamma+\nu_1^\ell},$$

where $\nu_1^\ell = \frac{\partial p_{1,-i}^\ell}{\partial x_1^i}$ is investor i's price impact in the local market. As before, the price impact of the dealer and of the investors in the local market ℓ are equilibrium objects that solve the following system of equations

$$\nu_1^\ell = \frac{1}{\frac{1}{\gamma+\lambda_1^\ell}+\frac{n^\ell-1}{\gamma+\nu_1^\ell}} \quad \text{and} \quad \lambda_1^\ell = \frac{1}{\frac{n^\ell}{\gamma+\nu_1^\ell}},$$

[6]Since the case of no interdealer market is a special case of the general model, and to avoid burdening the reader, we keep the same notation as above.

which yields the solution

$$\nu_1^\ell = \lambda_1^\ell = \frac{\gamma}{n^\ell - 1}. \tag{29}$$

The model without an interdealer market is equivalent to taking the limit of the model presented in Section 2 when the price impact in the interdealer market goes to infinity, i.e., $\lambda_2^\ell \to \infty$ for all $\ell \in N_D$. In this case, since dealers no longer access an interdealer market, the marginal value of the asset for each dealer ℓ is simply the expected marginal benefit of increasing the quantity that the dealer acquires in the local market and is given by

$$\lim_{\lambda_2^\ell \to \infty} \frac{dV_2^\ell}{dq_1^\ell} = \overline{w}^\ell \frac{\sum_{i \in N_I(\ell)} \theta^i}{n^\ell}$$

where $\overline{w}^\ell \equiv \lim_{\lambda_2^\ell \to \infty} w^\ell = \frac{\gamma}{\gamma + 2\lambda_1^\ell}$ and $\frac{dV_2^\ell}{dq_1^\ell}$ is given by Eq. 22. The investor i's value function given by Eq. (27) becomes

$$V_1^i(m) = \frac{\frac{\gamma}{2} + \nu_1^\ell}{\left(\gamma + \nu_1^\ell\right)^2} \mathbb{E}\left[\left(\frac{1}{2}\left(\theta^i - \frac{\sum_{j \in N_I(\ell)} \theta^j}{n^\ell}\right) + \frac{1}{2}\left(\theta^i - \overline{w}^\ell \frac{\sum_{j \in N_I(\ell)} \theta^j}{n^\ell}\right)\right)^2\right].$$

The market formation stage is just as in the setup with the interdealer market, described in Section 3.3. That is, an investor in a symmetric market structure, m_{n_S}, weighs the benefit of trading in a local market ℓ against one dealer and other $(n_S - 1)$ investors, against the benefit of trading in a local market ℓ' against one dealer and other n_S investors. We obtain the following result.

Proposition 4. ***(No interdealer market)*** *There exists a threshold $\hat{\rho}(n_S) \in [0,1)$ such that for all $\rho > \hat{\rho}(n_S)$ a fragmented symmetric market structure with n_D dealers and n_S investors is an equilibrium.*

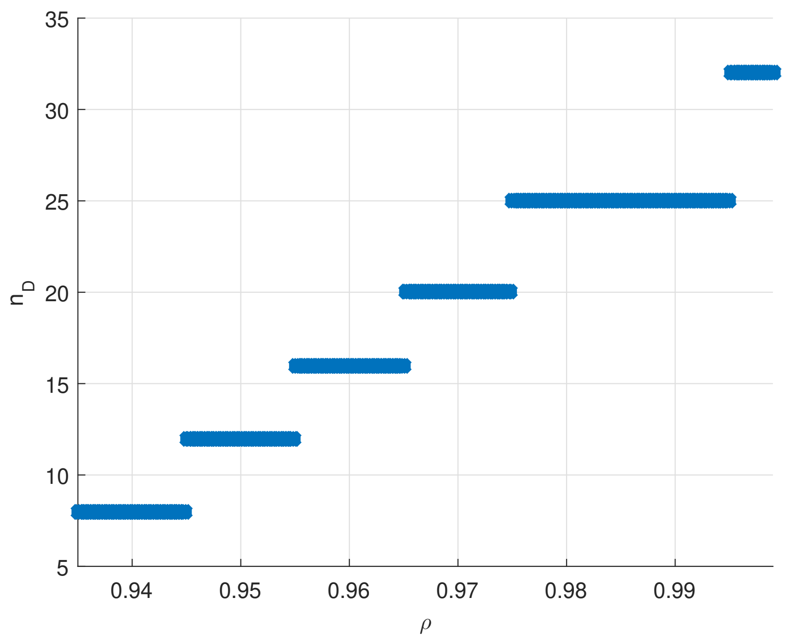
Proposition 4 shows that fragmentation can be supported in equilibrium even in the absence of an interdealer market. The intuition is similar to the one behind our main result. In fact, since the interdealer market provides a dealer the opportunity to re-trade, it only increases her willingness to intermediate. Thus, the presence of the interdealer market weakens the incentives of investors to trade in fragmented markets as it makes the dealer's demand more elastic to the size of their local market and decreases the loss in gains from trade with the dealer. Therefore, an interdealer market is not necessary to support market fragmentation in equilibrium. However, the role of dealers' strategic trading behavior can only be examined if we allow dealers to intermediate through the interdealer market. Moreover, an interdealer market is a prevalent features in the markets for some assets, as we argue in the introduction.

4.3 The maximum degree of fragmentation

So far, we have shown when a fragmented market can be supported in equilibrium, and that a key determinant of market fragmentation is dealers' strategic behavior. However, different markets have different degrees of fragmentation. In our set-up, the degree of fragmentation is determined by the number of dealers n_D that provide intermediation services. In particular, by keeping the number of investors, n_I, fixed and varying the number of dealers, n_D, (thus, changing the size of each local market) we can obtain the maximum degree of market fragmentation that can be supported in equilibrium for a given level of disagreement $(1-\rho)$. If we vary n_D while keeping n_I fixed then the number of investors need not be proportional with the number of dealers, as we consider in the main set-up. Thus, a strictly symmetric market structure, one in which all dealers have the same number of investors, does not always exist. To address the non-divisibility of investors we will focus on market structures in which the dispersion in the distribution of investors across dealers is minimized. We refer to these market structures as generalized symmetric.

Definition 2. A generalized symmetric market structure, m_G, with n_I investors and n_D dealers is a market structure in which x dealers have $\left\lfloor \frac{n_I}{n_D} \right\rfloor$ investors and $n_D - x$ have $\left\lfloor \frac{n_I}{n_D} \right\rfloor + 1$, where

$$x = n_D \left(\left\lfloor \frac{n_I}{n_D} \right\rfloor + 1 \right) - n_I.$$



Note: Figure 3 shows the maximum number of active dealers in a generalized symmetric market structure as a function of the correlation in investor's priors, ρ, for $n_I = 100$, $\gamma = 1$, and $\sigma_\theta^2 = 1$.

Figure 3: Maximum degree of fragmentation

Figure 3 illustrates the maximum degree of fragmentation of a generalized symmetric market that can be supported in equilibrium as a function of the correlation between investors' priors. As the figure shows, the maximum degree of fragmentation is higher when the disagreement between investors is lower. The intuition is straightforward. The higher the correlation between investors' priors, the more attractive it is for investors to trade in small local markets. Thus, when ρ is higher a market structure with more smaller local markets can be supported in equilibrium.

5 Welfare and liquidity in fragmented markets

In this section we analyze the welfare and liquidity properties of the equilibrium when markets are fragmented. To provide a meaningful characterization, we study the welfare and liquidity relative to the most natural benchmark of a centralized market.

5.1 Centralized market

A useful benchmark to study the implications of market fragmentation is a centralized market. We consider that in a centralized market structure, m_c, trade takes place between all investors and dealers only at date 1. Just as in the case of fragmented markets, the agents' trading strategies in a centralized market are represented by price-quantity schedules. In particular, the strategy of an agent is a map from her information set to the space of demand functions, as follows. The demand function of an investor i with prior θ^i is a continuous function $X_c^i : \mathbb{R} \to \mathbb{R}$, which maps the price p_1^c that prevails in the centralized market into a quantity x_c^i she wishes to trade. Similarly, the demand function of dealer ℓ who observes the common component θ is a continuous function $Q_c^\ell : \mathbb{R} \to \mathbb{R}$, which maps the price p^c, into a quantity q_c^ℓ she wishes to trade. The expected payoff for an investor i at date 0, corresponding to the strategy profile $\left\{X_c^i, Q_c^\ell\right\}_{i \in N_I, \ell \in N_D}$ is

$$V_c^i\left(m_c\right) = \mathbb{E}_0\left[\theta^i X_c^i\left(p_c;\theta^i\right) - \frac{\gamma}{2}\left(X_c^i\left(p_c;\theta^i\right)\right)^2 - p_c X_c^i\left(p_c;\theta^i\right)\right]$$

while the expected payoff of a dealer ℓ at date 0 is

$$V_c^\ell\left(m_c\right) = \mathbb{E}_0\left[-\frac{\gamma}{2}\left(Q_c^\ell\left(p_c;\theta\right)\right)^2 - p_c Q_c^\ell\left(p_c;\theta\right)\right],$$

where each p_c is the price at which the market clears, and it is given by

$$\sum_{i \in N_I} X_c^i\left(p_c;\theta^i\right) + \sum_{\ell \in N_D} Q_c^\ell\left(p_c;\theta\right) = 0.$$

In a centralized market, all of the $n_I + n_D$ market participants solve the same problem as the one solved by an investor in a fragmented market structure.

Proposition 5. ***(Existence and uniqueness)*** *Given a centralized market structure m_c there exists a unique symmetric equilibrium in linear strategies at date* 1*.*

As Proposition 5 shows, there is a unique equilibrium at date 1 in a centralized market structure. In this equilibrium, the demand function for an investor i is given by

$$X_c^i\left(p_c;\theta^i\right) = \frac{\theta^i - p_c}{\gamma + \lambda_c^i}, \tag{30}$$

where $\lambda_c^i = \frac{\partial p_{c,-i}}{\partial x_c^i}$ is the price impact of an investors in the centralized market. The demand function for a dealer ℓ is given by

$$Q_c^\ell\left(p_c;\theta\right) = -\frac{p_c}{\gamma + \lambda_c^\ell}, \tag{31}$$

where $\lambda_c^\ell = \frac{\partial p_{c,-\ell}}{\partial q_c^\ell}$ is the price impact of a dealer in the centralized market. In equilibrium, the price impacts are given by

$$\lambda_c^i = \lambda_c^\ell = \frac{\gamma}{n_I + n_D - 2},$$

and the price is

$$p_c = \frac{\sum_{j \in N_I} \theta^j}{n_I + n_D}. \tag{32}$$

As in the interdealer market, the equilibrium price in Eq. (32) in the centralized market is equal to the average priors of the market's participants. Putting together Eq. (30) and Eq. (32) we can see that a market participant h will hold a larger position the larger the difference between her prior θ^h and the average prior in the market, where $\theta^h = 0$ for dealers.

5.2 Welfare

In this section, we compare the investor and dealer welfare in fragmented and centralized market structures. For all the numerical illustrations in this section, we consider the case with $n_D = 7$, $n_S = 10$, $\gamma = 1$, and $\sigma_\theta^2 = 1$.

Investor welfare

The expected welfare of an investor i who chooses a dealer ℓ in a symmetric market structure m_{n_S} when the level of disagreement is $1 - \rho$ is given by

$$V_1^i\left(m_{n_S}\right) = \mathbb{E}_0\left[\theta^i x_1^i - \frac{\gamma}{2}\left(x_1^i\right)^2 - p_1^\ell x_1^i\right],$$

where x_1^i and p_1^ℓ are, respectively, the quantity of the asset the investor purchases and the price she pays for this quantity in equilibrium. The investor's expected welfare in a centralized market is

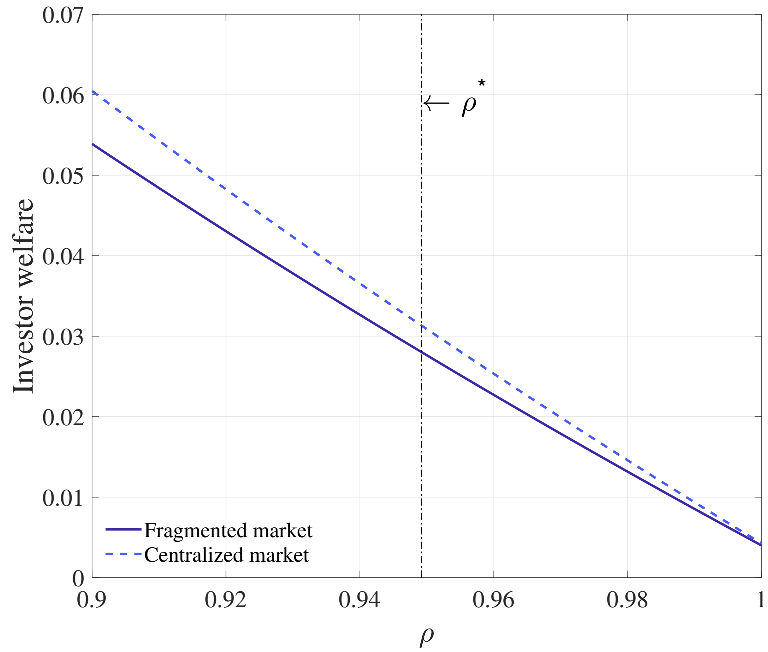
$$V_c^i\left(m_c\right) = \mathbb{E}_0\left[\theta^i x_c^i - \frac{\gamma}{2}\left(x_c^i\right)^2 - p_c x_c^i\right],$$

where x_c^i and p_c are, respectively, the quantity of the asset the investor purchases and the price she pays for this quantity in equilibrium.

Lemma 2. ***(Gains from trade)*** *For a given symmetric fragmented market structure m_{n_S}, an investor's welfare is continuous and monotonically increasing in the level of disagreement $(1 - \rho)$, i.e., $\frac{\partial V_1^i\left(m_{n_S}\right)}{\partial \rho} < 0$.*

Lemma 2 shows that the lower the disagreement in the market, the lower the investors' expected welfare in a symmetric fragmented market. A higher correlation in investor priors implies lower gains from trade between investors in the local markets and between dealers in the interdealer market. The lower these gains from trade, the lower the investors' expected welfare.

Let $\Delta V^i \equiv V_c^i\left(m_c\right) - V_1^i\left(m_{n_S}\right)$ be the difference between investors' expected utility in centralized and fragmented numbers with the same number of dealers. The following proposition compares investor welfare in centralized and fragmented markets.



Note: Figure 4 shows investor welfare in centralized and fragmented markets as a function of the correlation in investor priors, ρ. The solid line represents dealer welfare in a symmetric fragmented market structure and the dashed line represents dealer welfare in a centralized market structure. The dashed vertical line represents the threshold $\rho^{\star}$ above which a symmetric market structure is an equilibrium. The parameters for the figure are $n_D = 7$, $n_S = 10$, $\gamma = 1$, and $\sigma_\theta^2 = 1$.

Figure 4: Investor welfare

Proposition 6. ***(Investor welfare)*** *Investors are always better off in a centralized market structure, i.e,* $\Delta V^i > 0 \ \forall \rho$.

As Proposition 6 shows, investors are always better off in a centralized market structure than in a fragmented one. Figure 4 illustrates this result. The solid line represents investor welfare in fragmented markets, the dashed line represents investor welfare in centralized markets, and the vertical dotted line is the threshold $\rho^{\star}$ above which the symmetric fragmented market structure is an equilibrium. A centralized market structure reduces the investors' price impact and it increases the expected gains from trade. When all investors trade in the same market, the expected gains from trade for an investor are larger than when her trade with investors in other local markets is intermediated by dealers. Though the effect on the expected gains from trade disappears when $\rho = 1$, the lower price impact associated with trading in a bigger market is always present and investors are always strictly better off in a centralized market structure.

Dealer welfare

A dealer's expected welfare in a symmetric fragmented market structure is

$$V_1^{\ell}\left(m_{n_S}\right) = \mathbb{E}\left[-\frac{\gamma}{2}\left(q_1^{\ell} + q_2^{\ell}\right)^2 - p_1^{\ell} q_1^{\ell} - p_2^{\ell} q_2^{\ell}\right],$$

where q_1^ℓ and q_2^ℓ are, respectively, the quantities the dealer buys in the local market at price p_1^ℓ and in the interdealer market at price p_2, in equilibrium. In a centralized market structure, a dealer's expected welfare is

$$V_c^\ell(m_c) = \mathbb{E}\left[-\frac{\gamma}{2}\left(q_c^\ell\right)^2 - p_c q_c^\ell\right],$$

where q_c^ℓ is the quantity the dealer purchases at price p_c in the market, in equilibrium.

Lemma 3. ***(Gains from intermediation)*** *For a given symmetric fragmented market structure m_{n_S}, a dealer's welfare is continuous and monotonically decreasing in ρ, i.e., $\frac{\partial V_1^\ell(m_{n_S})}{\partial \rho} < 0$.*

Lemma 3 states that dealer welfare is increasing in the level of disagreement in the market, $(1-\rho)$. A higher correlation in investors' priors leads to a smaller price dispersion among the local markets. The only difference among dealers in their local markets in a symmetric market structure is the local price they face. Therefore, the more similar the prices in the local markets, the more similar the inventories dealers carry to the interdealer market and the smaller the gains for dealers from intermediating trade between the local markets through the interdealer market.

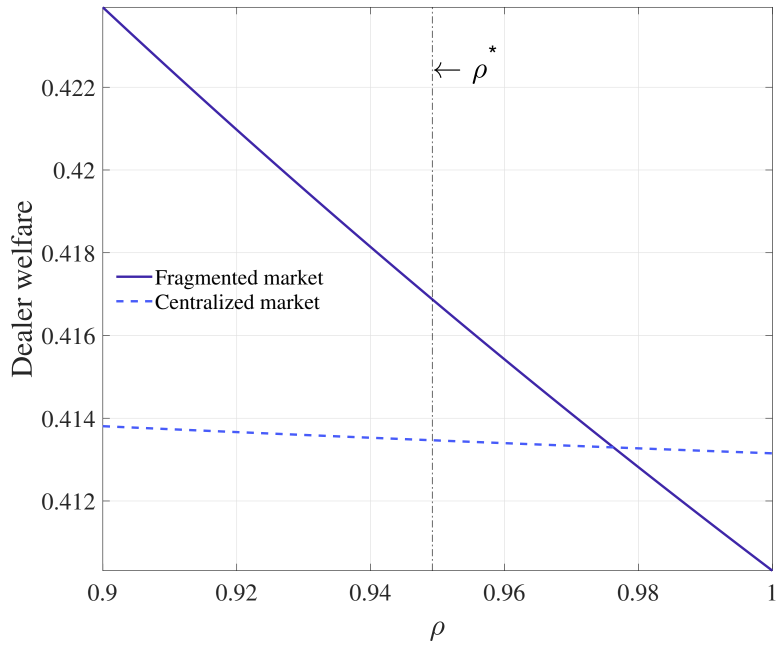
Let $\Delta V^D \equiv V_c^\ell(m_c) - V_1^\ell(m_{n_S})$ be the welfare gain for dealers from trading in a centralized market. The following proposition compares dealer welfare in centralized and fragmented markets.

Proposition 7. ***(Dealer welfare)*** *There exists a $\rho_W \in (0,1)$ such that for $\rho < \rho_W$ dealers are better off in symmetric fragmented markets and for $\rho > \rho_W$ dealers are better off in centralized markets, i.e.,*

$$\Delta V^D \begin{cases} > 0 & \text{if} \quad \rho > \rho_W \\ = 0 & \text{if} \quad \rho = \rho_W \\ < 0 & \text{if} \quad \rho < \rho_W \end{cases}.$$

Proposition 7 shows that dealers benefit from trading in a fragmented market structure when disagreement among investors is high and in a centralized market when disagreement is low. Trading in a fragmented market structure allows dealers to profit from intermediating trades between investors in different local markets through the interdealer market. The profits from intermediation disappear when the market structure is centralized. However, a centralized market offers a lower price impact than a fragmented one, making it cheaper for dealers to achieve their desired positions. When investor priors are very dispersed, $\rho < \rho_W$, the dealers' profits from intermediation are high and overcome the higher price impact associated with trading in fragmented market structures. In this case, dealers are better off in fragmented markets. When investor priors are very correlated and $\rho > \rho_W$, the dealers' profits from intermediating are small and the effect on price impact dominates. In this case, dealers are strictly better off trading in a centralized market.

Figure 5 illustrates a dealer's expected welfare in a fragmented market structure and in a centralized market structure. The solid line represents dealer welfare in fragmented markets which, consistent with Lemma 3, is decreasing in ρ. The dashed line represents dealer welfare in centralized markets and the vertical dotted line is the threshold $\rho^\star$ above which the symmetric fragmented market structure is an equilibrium. The threshold ρ_W above which dealers are better in a centralized market is given by the intersection of the dotted and solid lines.



Note: Figure 5 shows dealer welfare in centralized and fragmented markets as a function of the correlation in investor priors, ρ. The solid line represents dealer welfare in a symmetric fragmented market structure and the dashed line represents dealer welfare in a centralized market structure. The dashed vertical line represents the threshold $\rho^\star$ above which a symmetric market structure is an equilibrium. The parameters for the figure are $n_D = 7$, $n_S = 10$, $\gamma = 1$, and $\sigma_\theta^2 = 1$.

Figure 5: Dealer welfare

Efficiency

From Propositions 6 and 7 it follows that investors and dealers benefit differently from different market structures. When there is enough disagreement in the market, so that $\rho < \rho_W$, dealers are better off in fragmented markets, while investors are better off in centralized ones. If there is little disagreement, so that $\rho > \rho_W$, both investors and dealers are better off trading in a centralized market than trading in a fragmented one, and a fragmented symmetric market structure is inefficient. If $\rho^\star > \rho_W$, a fragmented market is inefficient even if it is supported in equilibrium. The inefficiency is due to a coordination failure which prevents investors from choosing to trade in a centralized market structure. Indeed, when $\rho > \rho^\star$, there is no benefit for an individual investor from deviating from a fragmented market structure m_{n_S} given that all other investors trade in the structure m_{n_S}. Naturally, our model abstracts from regulatory changes, that could have a major impact on how various assets are traded. However, if one thinks of regulatory reforms as a coordination device between market participants, centralization can arise when market fragmentation is inefficient. If this is the case, assets for which investors have very highly correlated priors could be traded in centralized markets.

5.3 Liquidity

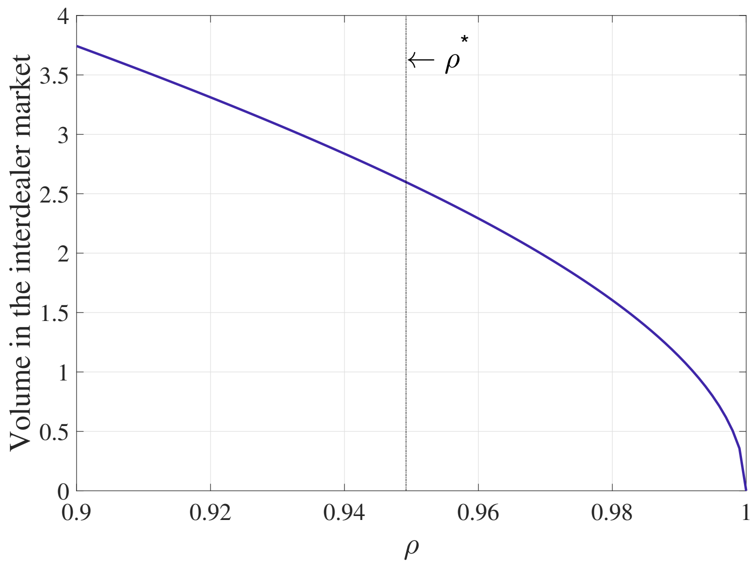
In this section, we study the implications of our model for liquidity. We start by looking at the liquidity of the interdealer market, which measures the amount of intermediation in the economy, and then focus on the liquidity of date 1 markets both in fragmented and centralized markets.

Intermediated volume

Dealers act as intermediaries between investors in different local markets through the interdealer market. The amount traded in the interdealer market is the intermediated volume in a fragmented market structure. The expected average trading volume in the interdealer market is

$$\mathcal{V}_D\left(m_{n_S}\right) = \frac{1}{2}\mathbb{E}\left[\frac{\sum_{\ell\in N_D}\left|q_2^\ell\right|}{n_D}\right].$$

As argued above, the dealers' willingness to intermediate decreases as the correlation in investors' priors approaches 1. Lemma 4 formalizes this intuition and shows that intermediated volume decreases with the correlation in investor priors and goes to zero as this correlation goes to 1.

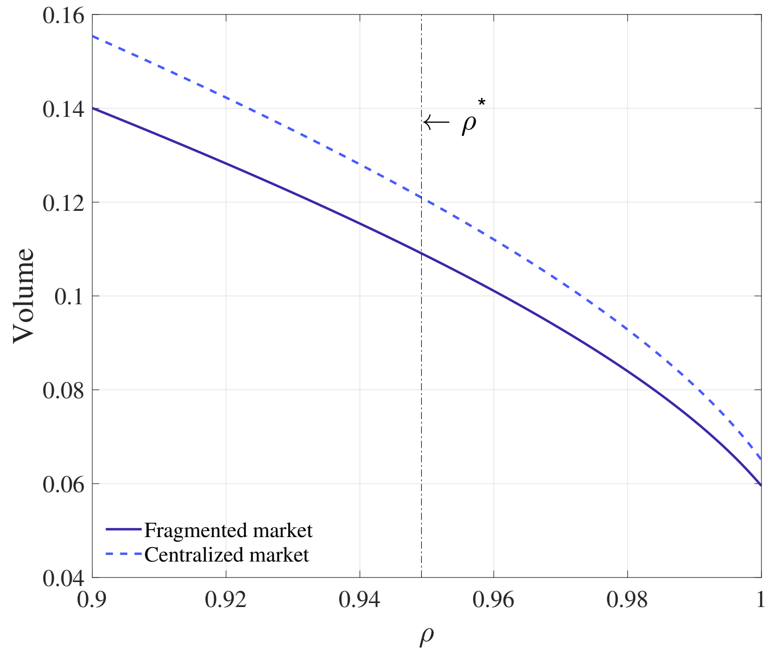


Note: Figure 6 shows traded volume in the interdealer market in a symmetric market structure as a function of the correlation in investor valuation, ρ. The dashed vertical line represents the threshold $\rho^\star$ above which a symmetric market structure is an equilibrium. The parameters for the figure are $n_D = 7$, $n_S = 10$, $\gamma = 1$, and $\sigma_\theta^2 = 1$.

Figure 6: Intermediated volume

Lemma 4. ***(Intermediated Volume)*** *In a symmetric fragmented market structure, the volume intermediated by dealers in the interdealer market increases with the level of disagreement* $(1-\rho)$*,* $\frac{\partial \mathcal{V}_D}{\partial \rho} < 0$ *with* $\lim_{\rho\to 1}\mathcal{V}_D = 0$.

When markets are fragmented, a higher correlation in investor priors leads to a smaller price dispersion in local markets. This smaller dispersion implies a smaller dispersion in dealer inventories and, thus, decreases the gains from trade in the interdealer market, which leads to a lower interdealer volume, as shown by Lemma 4. As the correlation in investor priors approaches 1, price dispersion



Note: Figure 7 shows the traded volume at date 1 in fragmented and centralized market structures as a function of the correlation in investor priors, ρ. The solid line represents aggregate volume in local markets in a symmetric fragmented market structure and the dashed line represents volume in a centralized market structure. The dashed vertical line represents the threshold $\rho^\star$ above which a symmetric market structure is an equilibrium. The parameters for the figure are $n_D = 7$, $n_S = 10$, $\gamma = 1$, and $\sigma_\theta^2 = 1$.

Figure 7: Volume

among local markets is 0 and intermediated volume goes to 0.[7] As we discuss in Section 5.2, when $\rho = 1$ and intermediated volume is zero, dealers do not profit from intermediation and they are better off in a centralized market than in a fragmented one. Figure 6 shows intermediated volume as a function of the correlation in investor priors ρ.

Volume

The liquidity of the markets at date 1 depends on the market structure. In a fragmented market structure, expected average trading volume in local market ℓ is given by

$$\mathcal{V}^\ell\left(m_{n_S}\right) = \frac{1}{2}\frac{\left|x_1^\ell\right| + \sum_{i\in N_I(\ell)}\left|q_1^i\right|}{n^\ell + 1}.$$

Analogously, in a centralized market structure, expected average trading volume is given by

$$\mathcal{V}_c = \frac{1}{2}\frac{\sum_{\ell\in N_D}\left|x_c^\ell\right| + \sum_{i\in N_I}\left|q_c^i\right|}{n_I + n_D}.$$

[7]Note that the observation that intermediated volume goes to 0 as ρ goes to 1 holds only in a symmetric fragmented market. Typically, trade in the interdealer market is positive if there is a different number of investors in each local market, even when $\rho = 1$.

The amount of trade in local markets also depends on the level of disagreement between investors. Lemma 5 shows that volume decreases with the correlation in investor priors in symmetric market structures, regardless of whether the market is fragmented.

Lemma 5. ***(Volume)*** *Expected trading volume in local markets increases with the level of disagreement in the market* $(1-\rho)$ *a) in the local markets in a fragmented symmetric market structure,* $\frac{\partial \mathcal{V}^\ell}{\partial \rho} < 0$*, and b) in a centralized market structure,* $\frac{\partial \mathcal{V}_c}{\partial \rho} < 0$ *.*

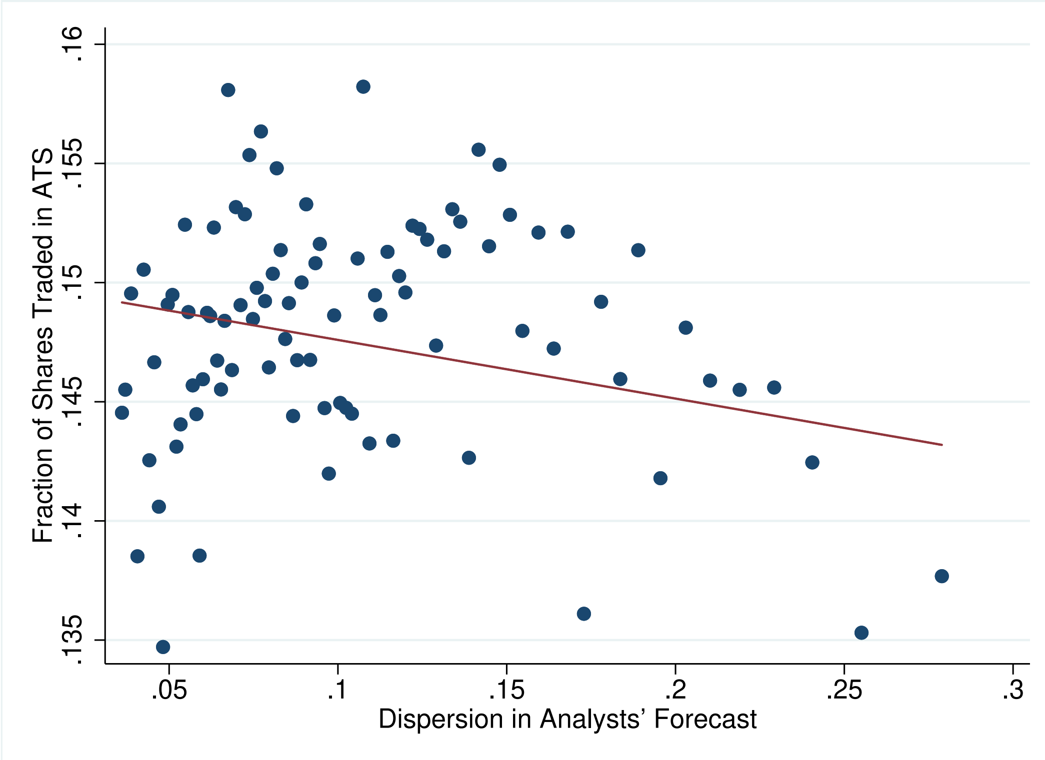
As can be seen from Lemma 5, the less disagreement among investors, the lower the gains from trade and, thus, the lower the incentives to trade in the market at date 1. Figure 7 depicts date 1 trading volume in centralized and fragmented symmetric market structures. The solid line represents the expected average volume in a fragmented market at date 1 measured as the average of the expected volume in all local markets, $\sum_{\ell \in N_D} \frac{\mathcal{V}^\ell}{n_D}$. The dashed line is the expected average volume in a centralized market and the dotted black line is the threshold $\rho^\star$ above which a symmetric fragmented market is an equilibrium. Lemma 5, together with Theorem 1, confirms the intuition that assets that are traded in fragmented markets have intrinsically low fundamental liquidity, proxied by high ρ. However, Figure 7 shows that the market structure itself also contributes to low volumes associated with decentralized markets. For a given level of disagreement $(1-\rho)$, the volume traded is lower in fragmented markets than in centralized ones.

6 Disagreement and fragmentation in equity markets

Our model suggests that an important feature in determining whether an asset is traded in a fragmented market is the disagreement between investors about the value of the asset. More specifically, our results imply that assets for which there is less disagreement are more likely to be traded in fragmented markets.

We investigate the relation between disagreement and market fragmentation, using data from equity markets. Typically, equity trading is not intermediated through interdealer markets. However, as Proposition 4 shows, in our model fragmentation can be supported in equilibrium even in the absence of an interdealer market provided there is little disagreement between investors. Thus, our results on market fragmentation can be applied to the context of equity markets.

We measure disagreement using the dispersion in analysts' forecasts from IBES price target data at the 12-month horizon. As in Diether, Malloy and Scherbina (2002), Yu (2011), and Hong and Sraer (2016) we measure disagreement as the monthly dispersion of analysts' forecasts. More specifically, we measure disagreement as the standard deviation of the analysts' forecasts normalized by the average analyst forecast for that stock in a given month. We have included only stocks for which we have more than three analyst forecasts. As a proxy for market fragmentation, we use the fraction of shares of a given stock that is traded in alternative trading systems (ATS) out of the total shares of the stocks traded in a given month. FINRA provides data about how many shares for each stock are traded in ATS, and we get the number of shares for each stock that is traded in main exchanges (NYSE, Nasdaq, and AMEX) from CRSP. Our measure of fragmentation is similar to the one used by O'Hara and Ye (2011). A more detailed description of the data and sample construction can be found in the Appendix.



Note: Figure 8 shows a bin-scatter plot of the relation between disagreement on the x-axis (measured as the dispersion in analysts' forecasts) and market fragmentation on the y-axis (measured as the fraction of shares of a given stock traded in ATS) in 2016.

Figure 8: Fragmentation and disagreement

Figure 8 shows, in a bin-scatter plot, the relation between our measure of disagreement, on the x-axis, and our measure of market fragmentation, on the y-axis, in 2016. We find a negative correlation between disagreement and market fragmentation. [8] This finding is consistent with our model's prediction in Section 4. In our model, the degree of market fragmentation is negatively associated with the level of disagreement between investors. A first look at the data suggests that the mechanism highlighted in our model is plausible and hints towards the importance of strategic trading in determining fragmented market structures. A more thorough analysis of the empirical relation between the degree of market fragmentation and disagreement, though interesting, is left for future work.

7 Conclusion

We develop a model of market formation in which investors have heterogeneous priors about the value of an asset to study the determinants of asset market fragmentation. When choosing a dealer, investors trade off the lower price impact that a larger market offers and a larger share of gains from trade with a dealer attained in a smaller market. When disagreement between investors is low, the decrease in gains

[8] Figure 8 is constructed using data winsorized at 5% and 95%. The negative correlation is robust to winsorizations at 2.5% and 1% , and to using unwinsorized data.

from trade with the dealer dominates the decrease in price impact, and investors have no incentives to deviate from a fragmented market structure. We find that dealers can benefit from trading in fragmented asset markets, while investors are always better off in centralized ones. When the correlation in investor priors is high enough, equilibrium fragmented markets are inefficient. Fragmented markets contribute to lower trading volume relative to a centralized markets. Our model emphasizes the role of investors' strategic behavior in determining the market structure in which an asset is traded. By focusing on the investors' choices, we add to the current view on the optimal market structure, which often considers that investors are passive player in the emergence of the market structure.

There are several important mechanisms that can contribute to markets having different degrees of fragmentation. Some of the mechanisms that have been proposed focus on information considerations through order flows, regulation (e.g., Reg ATS), technological advancements (e.g., automation, electronic LOBs), and minimum tick size (e.g., fees make effective prices almost continuous). In this paper, we focus on disagreement as the driving force for market fragmentation through the lens of a model that explores the interaction between two liquidity proxies: the price impact of the market participants and the amount of intermediation that dealers are willing to offer. Looking at these features allows us to identify a novel mechanism that contributes to the determination of the market structure. This mechanism is present even in the absence of differences in fee schedules, informational asymmetries or regulatory constraints. Though we find suggestive evidence consistent with our model's results, it remains an open empirical question to quantify the magnitudes of these competing forces.

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Appendix

A Proofs

Proof of Proposition 1

The first order condition in Eq. (12) for the dealer's optimization problem yields

$$Q_2^\ell\left(p_2;\theta,q_1^\ell\right)=\frac{-\gamma q_1^\ell-p_2}{\gamma+\lambda_2^\ell}$$

where $\lambda_2^\ell=\frac{\partial p_{2,-\ell}}{\partial q_2^\ell}$. We conjecture and subsequently verify that each dealer's equilibrium strategy is a linear demand function as follows

$$Q_2^\ell\left(p_2;\theta,q_1^\ell\right)=a_D q_1^\ell+b_D p_2.$$

Market clearing implies

$$p_2=\frac{a_D\sum_{\ell=1}^{n_D}q_1^\ell}{n_D b_D},\ \text{and}\ \lambda_2^\ell=-\frac{1}{\left(n_D-1\right)b_D}. \tag{A.1}$$

Then, matching coefficients

$$a_D=-\frac{\gamma}{\gamma-\frac{1}{(n_D-1)b_D}},\text{and}\ b_D=-\frac{1}{\gamma-\frac{1}{(n_D-1)b_D}}. \tag{A.2}$$

The unique solution for the system in Eq. (A.2) is given by

$$a_D=-\frac{n_D-2}{n_D-1},\text{and}\ b_D=-\frac{1}{\gamma}\frac{n_D-2}{n_D-1}.$$

Therefore, the equilibrium strategy of a dealer in the interdealer market is

$$Q_2^\ell\left(p_2;\theta,q_1^\ell\right)=-\frac{n_D-2}{n_D-1}\left(q_1^\ell+\frac{1}{\gamma}p_2\right)$$

and, using the market clearing condition, the equilibrium price is

$$p_2\left(\left\{q_1^l\right\}_{l\in N_D}\right)=-\gamma\frac{\sum_{l\in N_D}q_1^l}{n_D}.$$

Proof of Proposition 2

We conjecture and subsequently verify that in each local market ℓ the dealer's equilibrium strategy is a linear demand function given by

$$Q_1^\ell\left(p_1^\ell;\theta\right)=a^\ell\theta+b^\ell p_1^\ell,$$

and each investor's equilibrium strategy is a linear demand function given by

$$X_1^i\left(p_1^\ell;\theta^i\right)=\alpha^\ell\theta^i+\beta^\ell p_1^\ell.$$

Market clearing implies

$$p_1^\ell=-\frac{\left(n^\ell\alpha^\ell+a^\ell\right)\theta+\alpha^\ell\sum_{i\in N_I(\ell)}\eta^i}{n^\ell\beta^\ell+b^\ell},$$

and

$$\lambda_1^\ell=-\frac{1}{n^\ell\beta^\ell},\text{and}\ \upsilon_1^\ell=-\frac{1}{\left(n^\ell-1\right)\beta^\ell+b^\ell}. \tag{A.3}$$

The first order condition in Eq. (17) for the investor's optimization problem yields

$$x_1^i = \frac{\theta^i - p_1^\ell}{\gamma + \upsilon_1^\ell}, \tag{A.4}$$

where $\upsilon_1^\ell = \frac{\partial p_{1,-i}^\ell}{\partial x_1^i}$. The first order condition for the dealer's optimization problem is

$$\frac{dV_2^\ell(m)}{dq_1^\ell} - p_1^\ell - \frac{\partial p_{1,-\ell}^\ell}{\partial q_1^\ell} q_1^\ell = 0,$$

where $V_2^\ell(m)$ represents the payoff that dealer ℓ expects to receive after participating in the interdealer market and it is given by

$$V_2^\ell(m) = \mathbb{E}\left[-\frac{\gamma}{2}\left(q_2^{\ell\star} + q_1^\ell\right) - p_2 q_2^{\ell\star} \mid \theta, p_1^\ell\right].$$

Differentiating the profits of the dealer from participating in the interdealer market, we obtain

$$\frac{dV_2^\ell(m)}{dq_1^\ell} = \frac{\partial V_2^\ell(m)}{\partial q_1^\ell} + \frac{\partial V_2^\ell(m)}{\partial q_2^\ell}\frac{dq_2^\ell}{dq_1^\ell}.$$

Using the first order condition for the dealer in the interdealer market we have that

$$\frac{\partial V_2^\ell(m)}{\partial q_1^\ell} = 0,$$

which immediately implies that

$$\frac{dV_2^\ell(m)}{dq_1^\ell} = -\gamma\left(q_1^\ell + \mathbb{E}\left[Q_2^\ell\left(p_2; \theta, q_1^\ell\right) \mid \theta, q_1^\ell\right]\right).$$

Thus we can rewrite the first order condition for the dealer as

$$-\gamma\left(q_1^\ell + \mathbb{E}\left[Q_2^\ell\left(p_2; \theta, q_1^\ell\right) | \theta, q_1^\ell\right]\right) - p_1^\ell - \frac{\partial p_{1,-\ell}^\ell}{\partial q_1^\ell} q_1^\ell = 0.$$

Substituting the equilibrium demand function of the dealer in the interdealer market, $Q_2^\ell\left(p_2; \theta, q_1^\ell\right)$, we obtain that

$$q_1^\ell = \frac{\left(\gamma + \lambda_2^\ell\right)}{\gamma\lambda_2^\ell + \lambda_1^\ell\left(\gamma + \lambda_2^\ell\right)}\left(\frac{\gamma}{\gamma + \lambda_2^\ell}\mathbb{E}\left[p_2 \mid \theta, p_1^\ell\right] - p_1^\ell\right).$$

It follows that

$$q_1^\ell = \frac{-\gamma\frac{(n_D-2)}{n_D}\frac{\mathbb{E}\left[\sum_{l\in N_D, l\neq\ell} q_1^l|\theta\right]}{n_D-1} - p_1^\ell}{\gamma\frac{2}{n_D} + \frac{\partial p_1^\ell}{\partial q_1^\ell}}, \tag{A.5}$$

where

$$\mathbb{E}\left[\sum_{l\in N_D, l\neq\ell} q_1^l|\theta, q_1^\ell\right] = \mathbb{E}\left[\sum_{h\in N_D, h\neq\ell}\left(a^h\theta - b^h\left(\frac{\left(n^h\alpha^h + a^h\right)\theta + \alpha^h\sum_{i\in N_I(h)}\eta^i}{n^h\beta^h + b^h}\right)\right) \mid \theta\right]$$
$$= \sum_{h\in N_D, h\neq\ell}\left(a^h - b^h\frac{\left(n^h\alpha^h + a^h\right)}{n^h\beta^h + b^h}\right)\theta = \sum_{h\in N_D, h\neq\ell} n^h\frac{a^h\beta^h - \alpha^h b^h}{n^h\beta^h + b^h}\theta.$$

Using the demand functions in equations (A.4) and (A.5), and matching the coefficients with our guess for an equilibrium in linear strategies we get the following system

$$a^{\ell} = b^{\ell}\frac{n_D-2}{n_D}\frac{1}{n_D-1}\sum_{l\in N_D, l\neq \ell} n^l \frac{a^l\beta^l-\alpha^l b^l}{n^l\beta^l+b^l} \tag{A.6}$$
$$b^l = -\frac{n^{\ell}\beta^{\ell}n_D}{2\gamma n^{\ell}\beta^{\ell}-n_D}$$
$$\alpha^{\ell} = \frac{\left(n^{\ell}-1\right)\beta^{\ell}+b^{\ell}}{\gamma\left(\left(n^{\ell}-1\right)\beta^{\ell}+b^{\ell}\right)-1}$$
$$\beta^{\ell} = -\frac{\left(n^{\ell}-1\right)\beta^{\ell}+b^{\ell}}{\gamma\left(\left(n^{\ell}-1\right)\beta^{\ell}+b^{\ell}\right)-1}$$

We solve for β^{ℓ}in the system of equations (A.6). It follows that β^{ℓ} is given by the negative solution to

$$H(\beta)=0,$$

where

$$H(\beta) = -2n^{\ell}\left(n^{\ell}-1\right)(\gamma\beta)^2+\left(\left(2n^{\ell}-1\right)n_D+2n^{\ell}-2n^{\ell}\left(n^{\ell}-1\right)\right)\gamma\beta+2\left(n^{\ell}-1\right)n_D.$$

Since $H(0)>0$ and $H''(\cdot)<0$, a solution to $H(\beta)=0$ always exists and there is a unique negative root which determines β^{ℓ}. Then, $\left\{\alpha^{\ell},\beta^{\ell},b^{\ell}\right\}_{\ell\in ND}$ are uniquely determined. Let $\overrightarrow{a}=\left[a^1,a^2,...,a^{n_D}\right]'$. From (A.6) we have

$$\overrightarrow{a} = A\overrightarrow{a}+B,$$

where A is a $n_D\times n_D$ matrix where the element $A_{\ell l}=b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{n^l\beta^l}{n^l\beta^l+b^l}\in(-1,0)$ for all $l\neq\ell$, $l,\ell\in N_D$, $A^{\ell\ell}=0$ for all $\ell\in N_D$, and B is a n_D-dimensional column vector where the element

$$B_{\ell} = -b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\sum_{l\in N_D,l\neq\ell}n^l\frac{a^lb^l}{n^l\beta^l+b^l} \text{ for all } l\in N_D.$$

Then,

$$\left[I_{n_D}-A\right]\overrightarrow{a}=B,$$

where $[I_{n_D}-A]$ is invertible. Thus, $\overrightarrow{a}$ is uniquely determined.

Characterization of price sensitivities

The following two lemmas characterize the price sensitivities of investors and dealers in the local markets. We use these intermediate results in the proofs of the propositions in the main text.

Lemma 6. *(**Characterization of** β^{ℓ}) Investors' price sensitivity $\beta^{\ell}\in\left(-\frac{1}{\gamma},0\right)$ satisfies: a) $\frac{\partial\beta^{\ell}}{\partial n^{\ell}}<0$, b) $\frac{\partial\beta^{\ell}}{\partial n_D}<0$, c) $\bar{\beta}^{\ell}\equiv\lim_{n_D\to\infty}\beta^{\ell}\equiv-\frac{2\left(n^{\ell}-1\right)}{\gamma(2n^{\ell}-1)}$ and $\lim_{n\to\infty}\beta^{\ell}=-\frac{1}{\gamma}$.*

Proof. Recall that $\beta(n,n_D)$ is defined by the negative root of

$$H(\beta) = -2n(n-1)(\gamma\beta)^2+((2n-1)n_D-2n(n-2))\gamma\beta+2(n-1)n_D. \tag{A.7}$$

$\beta(n,n_D)$ is uniquely defined and $H'(\beta(n,n_D))>0$ since $H(0)>0$, and $H''<0$. Also, $\beta(n)>-\frac{1}{\gamma}$ since

$$H\left(-\frac{1}{\gamma}\right)=-2n-n_D<0.$$

a) Using the implicit function theorem we have

$$\frac{\partial\beta(n,n_D)}{\partial n}=-\frac{\frac{\partial H(\beta)}{\partial n}|_{\beta(n,n_D)}}{H'(\beta(n,n_D))}$$

and, since $H'(\beta(n,n_D))>0$,

$$sign\left(\frac{\partial\beta(n)}{\partial n}\right)=-sign\left(\frac{\partial H(\beta)}{\partial n}|_{\beta(n,n_D)}\right)=-sign(X),$$

where

$$\frac{\partial H(\beta)}{\partial n}|_{\beta(n,n_D)}=n_D-(2n-1)(\gamma\beta(n,n_D))^2+(n_D-2(n-1))\beta(n,n_D)\gamma \tag{A.8}$$

Since $H(\beta(n,n_D))=0$, using Eq. (A.7) we have

$$(\gamma\beta(n,n_D))^2=\frac{((2n-1)n_D-2n(n-2))\gamma\beta(n,n_D)+2(n-1)n_D}{2n(n-1)}.$$

Then, using this expression in Eq. (A.8) we have

$$\frac{\partial H(\beta)}{\partial n}|_{\beta(n,n_D)}=-\frac{1}{2n(n-1)}\left(\left(n_D(n-1)^2+(n_D+2)n^2\right)\gamma\beta(n,n_D)+2n_D(n-1)^2\right).$$

Then,

$$sign\left(\frac{\partial H(\beta)}{\partial n}|_{\beta(n,n_D)}\right)=\gamma\beta(n,n_D)-X,$$

where

$$X\equiv\frac{-2n_D(n-1)^2}{\left(n_D(n-1)^2+(n_D+2)n^2\right)}.$$

Note that $\gamma\beta(n)<X$ because

$$H(X)=\frac{2nn_D(n-1)\left(\left(2n^2-2n+1\right)n_D^2+4n\left(n^2-n+1\right)n_D+4n^2\left(n^2-2n+2\right)\right)}{\left(n_D(n-1)^2+(n_D+2)n^2\right)^2}>0$$

for all $n\geq1$ and for all $n_D\geq0$. Therefore, $\frac{\partial H(\beta)}{\partial n}|_{\beta(n,n_D)}<0$ which implies $\frac{\partial\beta(n,n_D)}{\partial n}<0$.

b) Also, using the implicit function theorem we have

$$\frac{\partial\beta(n,n_D)}{\partial n_D}=-\frac{\frac{\partial H(\beta)}{\partial n_D}|_{\beta(n,n_D)}}{H'(\beta(n,n_D))}$$

and

$$sign\left(\frac{\partial\beta(n,n_D)}{\partial n_D}\right)=-sign\left(\frac{\partial H(\beta)}{\partial n_D}|_{\beta(n,n_D)}\right),$$

where

$$\frac{\partial H(\beta)}{\partial n_D}|_{\beta(n,n_D)}=(2n-1)\gamma\beta(n,n_D)+2(n-1).$$

Note that $\gamma\beta(n)>-\frac{2(n-1)}{(2n-1)}$ because

$$H\left(-\frac{2(n-1)}{(2n-1)}\right)=-4\frac{n^2}{(2n-1)^2}(n-1)<0,$$

and, therefore, $\frac{\partial H(\beta)}{\partial n_D}|_{\beta(n)}>0$, which implies

$$\left(\frac{\partial\beta(n,n_D)}{\partial n_D}\right)<0.$$

c) Finally, from the definition of $\beta(n,n_D)$ we can write

$$\beta(n,n_D)=-\frac{1}{\gamma}\frac{1-\sqrt{1+\frac{16n(n-1)^2n_D}{((2n-1)n_D-2n(n-2))^2}}}{-\frac{4n(n-1)}{((2n-1)n_D-2n(n-2))}}=-\frac{1}{\gamma}\frac{\frac{4(n-1)n_D}{((2n-1)n_D-2n(n-2))}}{\left(1+\sqrt{1+\frac{16n(n-1)^2n_D}{((2n-1)n_D-2n(n-2))^2}}\right)}.$$

Taking limits gives

$$\lim_{n_D\to\infty}\beta(n,n_D)=-\frac{2(n-1)}{\gamma(2n-1)}\equiv\bar{\beta}\text{ and }\lim_{n\to\infty}\beta(n,n_D)=-\frac{1}{\gamma}.$$

□

Lemma 7. *(**Characterization of** b^ℓ) Dealers' price sensitivity b^ℓ satisfy a) $\frac{\partial b^\ell}{\partial n^\ell} < 0$, b) $\frac{\partial b^\ell}{\partial n_D} < 0$, and c) $\lim_{n_D\to\infty} b^\ell = -\frac{2n^\ell(n^\ell-1)}{\gamma(2n^\ell-1)}$ and $\lim_{n^\ell\to\infty} b^\ell = -\frac{n_D}{2\gamma}$.*

Proof. Using the definition of b^ℓ in Eq. (A.6) and Lemma 6 we have that

$$\frac{\partial b^\ell}{\partial n^\ell} = \frac{n_D^2}{(2\gamma n^\ell \beta^\ell - n_D)^2}\left(n^\ell \frac{\partial \beta^\ell}{\partial n^\ell} + \beta^\ell\right) < 0$$

and

$$\frac{\partial b^\ell}{\partial n_D} = \frac{-2\left(n^\ell \beta^\ell\right)^2 \gamma + n^\ell n_D^2 \frac{\partial \beta^\ell}{\partial n_D}}{\left(n_D - 2n^\ell \beta^\ell \gamma\right)^2} < 0,$$

which proves a) and b) in the lemma above.

Moreover, it also follows that

$$\lim_{n_D\to\infty} b^\ell = \lim_{n_D\to\infty} -\frac{n^\ell \beta^\ell}{\frac{2\gamma n^\ell \beta^\ell}{n_D} - 1} = n^\ell \bar{\beta}^\ell = -\frac{2n^\ell\left(n^\ell - 1\right)}{\gamma\left(2n^\ell - 1\right)}$$

and

$$\lim_{n^\ell\to\infty} b^\ell = \lim_{n^\ell\to\infty} -\frac{\beta^\ell n_D}{2\gamma\beta^\ell - \frac{n_D}{n^\ell}} = -\frac{n_D}{2\gamma},$$

which proves the statement in c). □

Proof of Lemma 1

Consider a local market ℓ. The first order condition in Eq. (17) for the investor's optimization problem yields

$$X_1^i\left(p_1^\ell;\theta^i\right) = \frac{\theta^i - p_1^\ell}{\gamma + \upsilon_1^\ell}.$$

Using that the inverse residual demand function of dealer ℓ in her local market, $p_{1,-\ell}^\ell$, is implied by

$$\sum_{j\in N_I(\ell)} X_1^j\left(p_1^\ell;\theta^i\right) + q_1^\ell = 0,$$

we obtain that her price impact, λ_1^ℓ, must satisfy the following equation

$$-\frac{n^\ell}{\gamma + \upsilon_1^\ell}\frac{\partial p_{1,-\ell}^\ell}{\partial q_1^\ell} + 1 = 0,$$

or

$$\lambda_1^\ell = \frac{\gamma + \upsilon_1^\ell}{n^\ell}.$$

Similarly, using that the inverse residual demand function for investor i in market ℓ, $p_{1,-i}^\ell$, is implied by

$$\sum_{j\in N_I(\ell), j\neq i} X_1^j\left(p_1^\ell;\theta^i\right) + x_1^i + Q_1^\ell\left(p_1^\ell;\theta\right) = 0,$$

we obtain that his price impact, υ_1^ℓ, must satisfy the following equation

$$-\frac{n^\ell - 1}{\gamma + \upsilon_1^\ell}\frac{\partial p_{1,-i}^\ell}{\partial x_1^i} + 1 + \frac{\partial Q_1^\ell}{\partial x_1^i} = 0. \tag{A.9}$$

To evaluate $\frac{\partial Q_1^\ell}{\partial x_1^i}$ we need to determine the indirect effect of a change in x_1^i on the expected price in the interdealer market. For this, we express the expected price in the interdealer market as a function of the price in the local market p_1^ℓ. We proceed as follows.

As we show in the proof of Proposition 2, the first order condition in Eq. (18) for the dealer's optimization problem implies that

$$Q_1^\ell\left(p_1^\ell;\theta\right)=\frac{\left(\gamma+\lambda_2^\ell\right)}{\gamma\lambda_2^\ell+\lambda_1^\ell\left(\gamma+\lambda_2^\ell\right)}\left(\frac{\gamma}{\gamma+\lambda_2^\ell}\mathbb{E}\left[p_2\mid\theta,p_1^\ell\right]-p_1^\ell\right). \tag{A.10}$$

Therefore, the market clearing condition in Eq. (5) becomes

$$\frac{\left(\gamma+\lambda_2^\ell\right)}{\gamma\lambda_2^\ell+\lambda_1^\ell\left(\gamma+\lambda_2^\ell\right)}\left(\frac{\gamma}{\gamma+\lambda_2^\ell}\mathbb{E}\left[p_2\mid\theta,p_1^\ell\right]-p_1^\ell\right)-\frac{1}{\upsilon_1^\ell+\gamma}\sum_{i\in N_I(\ell)}\left(\theta^i-p_1^\ell\right)=0,$$

which implies that the price in the local market ℓ is

$$p_1^\ell=\frac{1}{\frac{\left(\gamma+\lambda_2^\ell\right)}{\gamma\lambda_2^\ell+\lambda_1^\ell\left(\gamma+\lambda_2^\ell\right)}-\frac{n^\ell}{\gamma+\upsilon_1^\ell}}\left(\frac{\gamma}{\gamma\lambda_2^\ell+\lambda_1^\ell\left(\gamma+\lambda_2^\ell\right)}\mathbb{E}\left[p_2\mid\theta,p_1^\ell\right]-\frac{n^\ell}{\gamma+\upsilon_1^\ell}\frac{\sum_{i\in N_I(\ell)}\theta^i}{n^\ell}\right).$$

Further, the price in the interdealer market is

$$p_2=-\gamma\frac{\sum_{l\in N_D}q_1^l}{n_D}$$

and, substituting Eq. (A.10), we obtain

$$p_2=-\frac{\gamma}{n_D}\sum_{l\in N_D}\frac{\left(\gamma+\lambda_2^l\right)}{\gamma\lambda_2^l+\lambda_1^l\left(\gamma+\lambda_2^l\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left[p_2\mid\theta,p_1^l\right]-p_1^l\right).$$

Taking expectations, we have that

$$\mathbb{E}\left[p_2\mid\theta\right]=-\frac{\gamma}{n_D}\sum_{l\in N_D}\frac{\left(\gamma+\lambda_2^l\right)}{\gamma\lambda_2^l+\lambda_1^l\left(\gamma+\lambda_2^l\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left[\mathbb{E}\left[p_2\mid\theta,p_1^l\right]\mid\theta\right]-\mathbb{E}\left[p_1^l\mid\theta\right]\right),$$

and

$$\mathbb{E}\left[p_1^l\mid\theta\right]=\frac{1}{\frac{\left(\gamma+\lambda_2^l\right)}{\gamma\lambda_2^l+\lambda_1^l\left(\gamma+\lambda_2^l\right)}-\frac{n^l}{\gamma+\upsilon_1^l}}\left(\frac{\gamma}{\gamma\lambda_2^l+\lambda_1^l\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left[\mathbb{E}\left[p_2\mid\theta,p_1^l\right]\mid\theta\right]-\frac{n^l}{\gamma+\upsilon_1^l}\theta\right).$$

It follows that

$$\frac{\left(\gamma+\lambda_2^l\right)}{\gamma\lambda_2^l+\lambda_1^l\left(\gamma+\lambda_2^l\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left[\mathbb{E}\left[p_2\mid\theta,p_1^l\right]\mid\theta\right]-\mathbb{E}\left[p_1^l\mid\theta\right]\right)=-\frac{n^l}{\gamma+\upsilon_1^l}\left(\mathbb{E}\left[p_1^l\mid\theta\right]-\theta\right), \tag{A.11}$$

which implies that

$$\mathbb{E}\left[p_2\mid\theta\right]=\frac{\gamma}{n_D}\sum_{l\in N_D}\frac{n^l}{\gamma+\upsilon_1^l}\left(\mathbb{E}\left[p_1^l\mid\theta\right]-\theta\right).$$

Further, using that $\mathbb{E}\left[\mathbb{E}\left[p_2\mid\theta,p_1^l\right]\mid\theta,p_1^\ell\right]=\mathbb{E}\left[\mathbb{E}\left[p_2\mid\theta,p_1^\ell\right]\mid\theta\right]$, we obtain

$$\begin{aligned}\mathbb{E}\left[p_2\mid\theta,p_1^\ell\right]=&-\frac{\gamma}{n_D}\sum_{l\in N_D,l\neq\ell}\frac{\left(\gamma+\lambda_2^l\right)}{\gamma\lambda_2^l+\lambda_1^l\left(\gamma+\lambda_2^l\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left[\mathbb{E}\left[p_2\mid\theta,p_1^l\right]\mid\theta\right]-\mathbb{E}\left[p_1^l\mid\theta\right]\right)\\&-\frac{\gamma}{n_D}\frac{\left(\gamma+\lambda_2^\ell\right)}{\gamma\lambda_2^\ell+\lambda_1^\ell\left(\gamma+\lambda_2^\ell\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^\ell\right)}\mathbb{E}\left[p_2\mid\theta,p_1^\ell\right]-p_1^\ell\right).\end{aligned}$$

Using Eq. (A.11), we have that

$$\mathbb{E}\left[p_2|\theta,p_1^\ell\right]=\frac{\gamma}{n_D}\sum_{\substack{l\in N_D\\ l\neq\ell}}\frac{n^l}{\upsilon_1^l+\gamma}\left(\mathbb{E}\left[p_1^l|\theta\right]-\theta\right)-\frac{\gamma}{n_D}\frac{\left(\gamma+\lambda_2^\ell\right)}{\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell}\left(\frac{\gamma}{\left(\gamma+\lambda_2^\ell\right)}\mathbb{E}\left[p_2|\theta,p_1^\ell\right]-p_1^\ell\right).$$

That gives us that

$$\mathbb{E}\left[p_2|\theta,p_1^\ell\right]=\frac{\frac{\gamma}{n_D}}{1+\gamma\frac{1}{n_D}\frac{\gamma}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)}}\sum_{l\in N_D,l\neq\ell}\frac{n^l}{\upsilon_1^l+\gamma}\left(\mathbb{E}\left[p_1^l|\theta\right]-\theta\right)+\frac{\frac{\gamma}{n_D}\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}p_1^\ell.$$

Substituting back into Eq. (A.10), we obtain that the demand function of a dealer in the local market is

$$Q_1^\ell\left(p_1^\ell;\theta\right)=\frac{\frac{\gamma^2}{n_D}}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}\sum_{l\in N_D,l\neq\ell}\frac{n^l}{\upsilon_1^l+\gamma}\left(\mathbb{E}\left[p_1^l|\theta\right]-\theta\right)-\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}p_1^\ell.$$

Thus, we obtain that

$$\frac{\partial Q_1^\ell}{\partial x_1^i}=-\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}\frac{\partial p_{1,-i}^\ell}{\partial x_1^i}.$$

Substituting this in Eq. (4) we obtain that

$$-\frac{n^\ell-1}{\gamma+\upsilon_1^\ell}\frac{\partial p_{1,-i}^\ell}{\partial x_1^i}+1-\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}\frac{\partial p_{1,-i}^\ell}{\partial x_1^i}=0.$$

Finally, using Lemmas 6 and 7 and the expressions for ν_1^ℓ and λ_1^ℓ in Eq. (A.3) we have that the price impacts satisfy the following properties

$$a)\frac{\partial\upsilon_1^\ell}{\partial n^\ell}=\left(\frac{1}{\left(n^\ell-1\right)\beta^\ell+b^\ell}\right)^2\left(\beta^\ell+\left(n^\ell-1\right)\frac{\partial\beta^\ell}{\partial n^\ell}+\frac{\partial b^\ell}{\partial n^\ell}\right)<0,$$
$$b)\frac{\partial\lambda_1^\ell}{\partial n^\ell}=\left(\frac{1}{n^\ell\beta^\ell}\right)^2\left(\beta^\ell+\frac{\partial\beta^\ell}{\partial n^\ell}n^\ell\right)<0,$$
$$c)\frac{\partial\upsilon_1^\ell}{\partial n_D}=\left(\frac{1}{\left(n^\ell-1\right)\beta^\ell+b^\ell}\right)^2\left(\left(n^\ell-1\right)\frac{\partial\beta^\ell}{\partial n_D}+\frac{\partial b^\ell}{\partial n_D}\right)<0,\quad\text{and}$$
$$d)\frac{\partial\lambda_1^\ell}{\partial n_D}=\frac{1}{n^\ell\left(\beta^\ell\right)^2}\frac{\partial\beta^\ell}{\partial n_D}<0.$$

Proof of Theorem 1

We prove the result in three steps. First, we show in Lemma 8 below that

$$\lim_{n_D\to\infty}\Delta^i\left(\rho=1;n_D\right)<0,\tag{A.12}$$

where Δ^i has been defined in Eq. (24). Second as we show in Lemma 9 below that

$$\lim_{n_D\to 3}\Delta^i\left(\rho=1;n_D\right)>0.\tag{A.13}$$

Lastly, we show in Lemma 10 that $\Delta\left(\rho;n_D\right)$ is monotonically increasing in ρ, i.e.,

$$\frac{\partial\Delta^i\left(\rho;n_D\right)}{\partial\rho}>0.\tag{A.14}$$

From Eqs. (A.13) and (A.14) it follows that $\exists\overline{\rho}$ such that for all $\rho>\overline{\rho}$, $\Delta\left(\rho;n_D=3\right)>0$. From Eqs. (A.12) and (A.14) it follows that for any ρ

$$\lim_{n_D\to\infty} \Delta^i(\rho; n_D) < 0,$$

and, in particular, for all $\rho > \bar{\rho}$. Therefore, since $\Delta^i(\rho; n_D)$ is continuous in n_D, for all $\rho^* > \bar{\rho}$ there exists $n_D(\rho^*)$ such that

$$\Delta^i(\rho^*; n_D(\rho^*)) = 0.$$

Using Eq. (A.14), this implies that for all $\rho > \rho^*$

$$\Delta^i(\rho; n_D(\rho^*)) > 0.$$

Next, we present the proofs of the aiding lemmas. For the remainder of the appendix, we denote by $\beta(n, n_D)$ is the negative root of $H(\beta; n, n_D) = 0$ as defined in Eq. (A.7). Similarly, $b(n, n_D)$ satisfies the system in Eqs. (A.6).

Lemma 8. *If investor valuations are perfectly correlated, an investor has incentives to deviate if the interdealer market is competitive, i.e.,*

$$\lim_{n_D\to\infty} \Delta^i(\rho = 1; n_D) < 0.$$

Proof. From the definition of Δ^i in Eq. (24) we have

$$\Delta^i(\rho = 1; n_D, n_S) = -\left(\frac{\beta^{sym}\left(1+\frac{\gamma}{2}\beta^{sym}\right)}{(n_S\beta^{sym}+b^{sym})^2}\left(b^{sym}+a^{sym}\right)^2 - \frac{\beta^{dev}\left(1+\frac{\gamma}{2}\beta^{dev}\right)}{((n_S+1)\beta^{dev}+b^{dev})^2}\left(b^{dev}+a^{dev}\right)^2\right)\sigma_\theta^2,$$

where

$$\beta^{sym} = \beta(n_S, n_D) \text{ and } \beta^{dev} = \beta(n_S+1, n_D),$$

and $a^{sym}$and $a^{dev}$are as defined in Eqs. (A.6) and (C.2), respectively. Then,

$$\lim_{n_D\to\infty} \Delta^i(1; n_D, n_S) = -\left(\frac{\bar{\beta}^{sym}\left(1+\frac{\gamma}{2}\bar{\beta}^{sym}\right)\left(n_S\bar{\beta}^{sym}+\lim_{n_D\to\infty} a^{sym}\right)^2}{\left(2n_S\bar{\beta}^{sym}\right)^2} - \frac{\beta^{dev}\left(1+\frac{\gamma}{2}\beta^{dev}\right)\left((n_S+1)\bar{\beta}^{dev}+\lim_{n_D\to\infty} a^{dev}\right)^2}{\left(2(n_S+1)\bar{\beta}^{dev}\right)^2}\right)\sigma_\theta^2.$$

Using Lemma 7 and Lemma 18 below we have

$$\lim_{n_D\to\infty} \Delta^i(1; n_D, n_S) = -\left(\frac{1}{2-\gamma\bar{\beta}^{sym}n_S}\right)^2\frac{4n_S}{\gamma\left(4n_S^2-1\right)^2}\sigma_\theta^2 < 0,$$

where we used that

$$\bar{\beta}^{sym} \equiv \lim_{n_D\to\infty} \beta(n_S) = -\frac{2(n_S-1)}{\gamma(2n_S-1)}, \text{ and } \bar{\beta}^{dev} \equiv \lim_{n_D\to\infty} \beta(n_S+1) = -\frac{2n_S}{\gamma(2n_S+1)}.$$

□

Lemma 9. *If investor valuations are perfectly correlated, an investor has no incentives to deviate if there are three active dealers, i.e.*

$$\lim_{n_D\to 3} \Delta^i(\rho = 1; n_D, n_S) > 0.$$

Proof. Let $b^{sym} \equiv b(n_S, n_D = 3)$, $b^{dev} \equiv b(n_S+1, n_D = 3)$ and $b^{\ell} \equiv b(n_S-1, n_D = 3)$. From the definition of Δ^i in Eq. (24) we have

$$
\begin{aligned}
&\lim_{n_D \to 3} \Delta^i \left(\rho = 1; n_D, n_S\right) \\
&= \lim_{n_D \to 3} - \left(\frac{\beta^{sym}\left(1 + \frac{\gamma}{2}\beta^{sym}\right)}{\left(n_S \beta^{sym} + b^{sym}\right)^2} \left(b^{sym} + a^{sym}\right)^2 - \frac{\beta^{dev}\left(1 + \frac{\gamma}{2}\beta^{dev}\right)}{\left(\left(n_S + 1\right)\beta^{dev} + b^{dev}\right)^2} \left(b^{dev} + a^{dev}\right)^2 \right) \sigma_\theta^2 \\
&= \lim_{n_D \to 3} - \left(\frac{\frac{3b^{sym}}{n_S(2\gamma b^{sym}+3)}\left(1+\frac{\gamma}{2}\frac{3b^{sym}}{n_S(2\gamma b^{sym}+3)}\right)}{\left(\frac{3b^{sym}}{(2\gamma b^{sym}+3)}+b^{sym}\right)^2} \left(2b^{sym}\frac{\gamma b^{sym}+3}{\gamma b^{sym}+6}\right)^2 - \frac{\beta^{dev}\left(1+\frac{\gamma}{2}\beta^{dev}\right)}{\left((n_S+1)\beta^{dev}+b^{dev}\right)^2} \left(b^{dev} + a^{dev}\right)^2 \right) \sigma_\theta^2 \\
&= \lim_{n_D \to 3} - \left(\frac{3}{2} \frac{b^{sym}(2n_S(2\gamma b^{sym}+3)+3\gamma b^{sym})}{(n^\ell)^2(\gamma b^{sym}+6)^2} - \frac{3}{2} \frac{b^{dev}\left(2\left(2\gamma b^{dev}+3\right)(n_S+1)+\gamma 3b^{dev}\right)}{(n_S+1)^2(\gamma b^{dev}+6)^2} \left(\frac{\left(\gamma b^{dev}+6\right)}{(2\gamma b^{dev}+6)} \left(1 + \frac{a^{dev}}{b^{dev}}\right)\right)^2 \right) \sigma_\theta^2,
\end{aligned}
$$

where we used that

$$
a^{sym} = \frac{\gamma \left(b^{sym}\right)^2}{\gamma b^{sym} + 6} \text{ and } n^\ell \beta^{sym} = \frac{3b^{sym}}{\left(2\gamma b^{sym} + 3\right)}.
$$

From Lemma 19 in the Online Appendix we know that

$$
-\frac{4\gamma b^{sym} n_S + 6n_S + \gamma 3b^{sym}}{n_S \left(\gamma b^{sym} + 6\right)^2} b^{sym} + \frac{4\gamma \left(n_S + 1\right) b^{dev} + 6\left(n_S + 1\right) + \gamma 3b^{dev}}{\left(n_S + 1\right)\left(\gamma b^{dev} + 6\right)^2} b^{dev} > 0 \tag{A.15}
$$

and from Lemma 20 in the Online Appendix we have

$$
\frac{n_S + 1}{n_S} > \left(\frac{\left(\gamma b^{dev} + 6\right)}{\left(2\gamma b^{dev} + 6\right)} \left(1 + \frac{a^{dev}}{b^{dev}}\right)\right)^2 > 0. \tag{A.16}
$$

Then, from Eq. (A.15) we have

$$
-\frac{4\gamma b^{sym} n_S + 6n_S + \gamma 3b^{sym}}{n_S^2 \left(\gamma b^{sym} + 6\right)^2} b^{sym}\left(n_S + 1\right) + \frac{4\gamma \left(n_S + 1\right) b^{dev} + 6\left(n_S + 1\right) + 3\gamma b^{dev}}{n_S \left(\gamma b^{dev} + 6\right)^2} b^{dev} > 0
$$

and using Eq. (A.16) it follows that

$$
\lim_{n_D \to 3} \Delta^i \left(\rho = 1; n_D, n_S\right) > 0.
$$

□

Lemma 10. Δ^i *is monotonically increasing in* ρ, *i.e.,* $\frac{\partial \Delta^i(\rho; n_D, n_S)}{\partial \rho} > 0$.

Proof. From the definition of Δ^i we have

$$
\frac{\partial \Delta^i \left(\rho; n_D, n_S\right)}{\partial \rho} = L\left(n_S + 1\right) - L\left(n_S\right),
$$

where

$$
\begin{aligned}
L\left(n\right) &\equiv -\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right) + 2\right)\left(\left(\frac{\left(b\left(n\right) + \left(n - 1\right)\beta\left(n\right)\right)}{\left(b\left(n\right) + n\beta\left(n\right)\right)}\right)^2 + \left(n - 1\right)\left(\frac{\beta\left(n\right)}{\left(b\left(n\right) + n\beta\left(n\right)\right)}\right)^2\right) \\
&= -\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right) + 2\right)\left(1 - \frac{\beta\left(n\right)}{\left(b\left(n\right) + n\beta\left(n\right)\right)} - \frac{\beta\left(n\right) b\left(n\right)}{\left(b\left(n\right) + n\beta\left(n\right)\right)^2}\right) > 0.
\end{aligned}
$$

Taking the derivative with respect to n we have

$$
\frac{\partial L\left(n\right)}{\partial n} > 0
$$

because from Lemma 21 in the Online Appendix we have that $1-\frac{\beta(n)}{(b(n)+n\beta(n))}$ is increasing in n, from Lemma 23 we know $-\frac{\beta(n)b(n)}{(b(n)+n\beta(n))^2}$ is increasing in n, and

$$\frac{\partial\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\right)}{\partial n}=-\left(\beta\left(n\right)\gamma+1\right)\frac{\partial\beta\left(n\right)}{\partial n}>0,$$

because $0>\beta\left(n\right)>-\frac{1}{\gamma}$ and $\frac{\partial\beta}{\partial n}<0$ from Lemma 6. Then, $L\left(n_S+1\right)-L\left(n_S\right)>0$. □

Proof of Proposition 3

The proof follows from Lemma 8 which shows that

$$\lim_{n_D\to\infty}\Delta^i\left(\rho=1;n_D,n_S\right)<0,$$

and Lemma 10 which shows that $\Delta^i\left(\rho;n_D,n_S\right)$ is monotonically increasing in ρ. This implies that

$$\Delta^i\left(\rho;n_D,n_S\right)<\Delta^i\left(\rho=1;n_D,n_S\right)$$

for any $\rho<1$. Taking the limit as $n_D\to\infty$, we obtain that

$$\lim_{n_D\to\infty}\Delta^i\left(\rho;n_D,n_S\right)<0.$$

Proof of Proposition 4

At date 0, when investor i chooses a dealer with whom to trade, her expected payoff in a symmetric market structure m_{n_s} with no interdealer market is given by

$$V_1^i\left(m_{n_s}\right)=\mathbb{E}_0\left[\theta^i x_1^i-\frac{\gamma}{2}\left(x_1^i\right)^2-p_1^{\ell}x_1^i\right],$$

where x_1^i is the quantity of the asset the investor purchases at the price p_1^{ℓ} when the investor's prior is θ^i and follows from Eq. (19) with $\upsilon_1^{\ell}=\frac{\gamma}{n_S-1}$. The equilibrium price p_1^{ℓ} is given by

$$p_1^{\ell}=\frac{\sum_{i\in N_I(\ell)}\theta^i}{n_S+1}.$$

Substituting the quantity that the investor trades in equilibrium into her expected payoff we obtain

$$V_1^i\left(m_{n_s}\right)=\frac{1}{2\gamma}\frac{\left(n_S\right)^2-1}{\left(n_S\right)^2}\mathbb{E}_0\left[\left(\theta^i-p_1^{\ell}\right)^2\right],$$

which can be written as

$$V_1^i\left(m_{n_s}\right)=\frac{1}{2\gamma}\left(\sigma_\theta^2+\sigma_\eta^2\right)\frac{n_S-1}{\left(n_S\right)^2}\frac{n_S\left(n_S+1\right)\left(1-\rho\right)+2\rho-1}{\left(n_S+1\right)}.$$

In the case of no interdealer market, a symmetric market structure m_{n_S} is stable if the expected payoff of an investor, $V_1^i\left(m_{n_S}\right)$, is decreasing in n_S. In other words, the market structure m_{n_S} is stable if

$$\frac{\partial}{\partial n_S}V_1^i\left(m_{n_s}\right)=-\frac{1}{n_S^3\left(n_S+1\right)^2}\left(\left(n_S^3+6n_S^2-3n_S-4\right)\rho+\left(n_S-4n_S^2-n_S^3+2\right)\right)\leq 0$$

or when

$$\rho\geq\frac{n_S^3+4n_S^2-n_S-2}{n_S^3+6n_S^2-3n_S-4}\equiv\hat{\rho}\left(n_S\right).$$

Note that

$$\frac{n_S^3+4n_S^2-n_S-2}{n_S^3+6n_S^2-3n_S-4}<1$$

for all $n_S\geq 2$.

Proof of Proposition 5

The first order condition for an investor $i \in N_I$ in a centralized market is given by

$$(\theta^i - p_c) - \left(\gamma + \frac{\partial p_{c,-i}}{\partial x_c^i}\right) x_c^i = 0,$$

where $p_{c,-i}$ is the inverse residual demand of investors i implied by

$$\sum_{j \in N_I, j \neq i} X_c^j \left(p_c; \theta^j\right) + \sum_{l \in N_D} Q_c^l \left(p_c; \theta\right) + x_c^i = 0.$$

The first order condition for a dealer $\ell \in N_D$ in a centralized market is given by

$$-p_c - \left(\gamma + \frac{\partial p_{c,-\ell}}{\partial q_c^\ell}\right) q_c^\ell = 0,$$

where $p_{c,-\ell}$ is the inverse residual demand of dealer ℓ implied by

$$\sum_{j \in N_I} X_c^j \left(p_c; \theta^j\right) + \sum_{l \in N_D, l \neq \ell} Q_c^l \left(p_c; \theta\right) + q_c^\ell = 0.$$

The demand functions for investors and dealers implied by these first order conditions are, respectively,

$$X_c^i \left(p_c; \theta^i\right) = \frac{\theta^i - p_c}{\gamma + \lambda_c^i} \quad \text{and} \quad Q_c^\ell \left(p_c; \theta\right) = -\frac{p_c}{\gamma + \lambda_c^\ell},$$

where $\lambda_c^i = \frac{\partial p_{c,-i}}{\partial x_c^i}$ and $\lambda_c^\ell = \frac{\partial p_{c,-\ell}}{\partial q_c^\ell}$ are the price impacts of investors and dealers in the centralized markets.

In an equilibrium in linear strategies we conjecture and subsequently verify that the demand functions of investors and dealers are given by

$$X_c^i \left(p_c; \theta^i\right) = \alpha^c \theta^i + \beta^c p_c \quad \text{and} \quad Q_c^\ell \left(p_c; \theta\right) = b^c p_c.$$

Matching coefficients we have

$$\alpha^c = -\beta^c = \frac{1}{\gamma + \lambda_c^i} \quad \text{and} \quad b^c = -\frac{1}{\gamma + \lambda_c^\ell}. \tag{A.17}$$

Market clearing implies that the equilibrium price in the centralized market is given by

$$p_c = -\frac{\alpha^c \sum_{j \in N_I} \theta^j}{n_I \beta^c + n_D b^c}$$

and the price impacts given by

$$\lambda_c^i = \frac{1}{(n_I - 1)\beta^c + n_D b^c} \quad \text{and} \quad \lambda_c^\ell = \frac{1}{n_I \beta^c + (n_D - 1) b^c}. \tag{A.18}$$

The equilibrium demand coefficients are given by the unique solution to the system in Eqs. (A.17) and (A.18)

$$\alpha^c = -\beta^c = -b^c = \frac{1}{\gamma} \frac{n_I + n_D - 2}{n_I + n_D - 1}.$$

In equilibrium, the price impacts are given by

$$\lambda_c^i = \lambda_c^\ell = \frac{\gamma}{n_I + n_D - 2}$$

and the price is $p_c = \frac{\sum_{j \in N_I} \theta^j}{n_I + n_D}$.

Proof of Lemma 2

The expected utility of an investor i in local market ℓ in a symmetric market structure is

$$V_1^i\left(m_{n_S};\rho\right)=\mathbb{E}\left[\theta^i x_1^i-\frac{\gamma}{2}\left(x_1^i\right)^2-p_1^\ell x_1^i\right], \tag{A.19}$$

where x_1^i and p_1^ℓ are the equilibrium quantity acquired by investor i and the equilibrium price in local market ℓ, respectively. In this case $n^\ell=n_S\ \forall\ell\in N_D$.

The equilibrium price in local market ℓ is

$$p_1^\ell=\pi_\theta\theta+\pi_\eta\sum_{i\in N^\ell}\eta^i,$$

where

$$\pi_\theta\equiv-\frac{n^\ell\alpha^\ell+a^\ell}{n^\ell\beta^\ell+b^\ell}\text{ and }\pi_\eta\equiv\frac{\beta^\ell}{n^\ell\beta^\ell+b^\ell}. \tag{A.20}$$

Using the equilibrium linear strategies, we have

$$\mathbb{E}\left[\theta^i x_1^i\right]=\mathbb{E}\left[\theta^i\left(\alpha^\ell\theta^i+\beta^\ell p_1^\ell\right)\right]=-\frac{\beta^\ell\left(a^\ell+b^\ell\right)}{n^\ell\beta^\ell+b^\ell}\sigma_\theta^2-\frac{\left(n^\ell-1\right)\beta^\ell+b^\ell}{n^\ell\beta^\ell+b^\ell}\beta^\ell\sigma_\eta^2$$

$$\begin{aligned}\mathbb{E}\left[\left(x_1^i\right)^2\right]&=\mathbb{E}\left[\left(\alpha^\ell\theta^i+\beta^\ell p_1^\ell\right)^2\right]\\&=\left(\frac{\beta^\ell}{b^\ell+n^\ell\beta^\ell}\left(a^\ell+b^\ell\right)\right)^2\sigma_\theta^2+\left(\left(\left(n^\ell-1\right)\beta^\ell+b^\ell\right)^2+\left(n^\ell-1\right)\beta^{\ell2}\right)\left(\frac{\beta^\ell}{n^\ell\beta^\ell+b^\ell}\right)\sigma_\eta^2,\end{aligned}$$

and

$$\mathbb{E}\left[p_1^\ell x_1^i\right]=\pi_\theta\left(\alpha^\ell+\beta^\ell\pi_\theta\right)\sigma_\theta^2+\pi_\eta\left(\alpha^\ell+\beta^\ell\pi_\eta\right)\sigma_\eta^2$$

Then, Eq. (A.19) becomes

$$V_1^i\left(m_n;\rho\right)=-\beta^\ell\left(\frac{\gamma}{2}\beta^\ell+1\right)\frac{1}{\left(n^\ell\beta^\ell+b^\ell\right)^2}\left(\left(a^\ell+b^\ell\right)^2+\left(\left(\left(n^\ell-1\right)\beta^\ell+b^\ell\right)^2+\left(n^\ell-1\right)\beta^{\ell2}\right)\frac{1-\rho}{\rho}\right)\sigma_\theta^2 \tag{A.21}$$

Since β^ℓ and b^ℓ do not depend on ρ, and $\beta^\ell\in\left(-\frac{1}{\gamma},0\right)$, we have that

$$\frac{\partial V_1^i\left(m;\rho\right)}{\partial\rho}=\beta^\ell\left(\frac{\gamma}{2}\beta^\ell+1\right)\frac{\left(\left(n^\ell-1\right)\beta^\ell+b^\ell\right)^2+\left(n^\ell-1\right)\beta^{\ell2}}{\left(n^\ell\beta^\ell+b^\ell\right)^2}\frac{1}{\rho^2}<0.$$

Proof of Proposition 6

Investor welfare can be rewritten as

$$V_1^i\left(m;\rho\right)=-\beta^\ell\left(\frac{\gamma}{2}\beta^\ell+1\right)\left(\left(1-\pi_\theta\right)^2+\left(\left(1-\pi_\eta\right)^2+\pi_\eta^2\left(n^\ell-1\right)\right)\frac{1-\rho}{\rho}\right)\sigma_\theta^2 \tag{A.22}$$

where π_θ and π_η are such that

$$p_1=\pi_\theta\theta+\pi_\eta\sum_{j\in N^\ell}\eta^j.$$

Let $\Delta V^i=\mathbb{E}\left(V^{i,c}\right)-\mathbb{E}\left(V^i\right)$ be the difference between an investor's welfare in a centralized market withn_D dealers and a symmetric fragmented market with n_D dealers. The proof follows directly from Lemma 12 and Lemma 13 below. These lemmas use the following intermediate result.

Lemma 11. ***(Price function coefficients)*** *The price function loading on the average idiosyncratic priors in a symmetric market structure is higher than that of a centralized market, i.e.,* $\pi_\eta^s>\pi_\eta^c$

Proof. Using the definition of the price coefficients in Eq. (A.20) we have

$$\pi_\eta^s > \pi_\eta^c$$
$$\frac{1}{n_S}\frac{n_D - 2\beta^s\gamma n_S}{2n_D - 2\beta^s\gamma n_S} > \frac{1}{(n_S+1)\,n_D}$$
$$\beta^s\gamma n_S < \frac{n_D\,(n_D-2)\,n_S + n_D^2}{2\left((n_D-1)\,n_S + n_D\right)}$$

which always holds since $\beta^s < 0$ and the right hand side of this expression is positive. □

Lemma 12. *Investors are better off in a centralized market structure as ρ approaches* 1*, i.e.,* $\lim_{\rho\to 1}\Delta V^i > 0$.

Proof. Using the definition of ΔV^i and taking limits as $\rho \to 1$ we have

$$\lim_{\rho\to 1}\Delta V^i = -\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\left(\frac{n_D}{n_I+n_D}\right)^2 - \left(-\beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\right)\left(-\frac{1}{\beta^s\gamma n_S - 2}\right)^2.$$

This limit can be rewritten as

$$\lim_{\rho\to 1}\Delta V^i = \left(\beta^s\left(\frac{\gamma}{2}\beta^s+1\right) - \beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\right)\left(-\frac{1}{\gamma\beta^s n_s - 2}\right)^2 - \beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\left(\left(\frac{n_D}{n_I+n_D}\right)^2 - \left(-\frac{1}{\gamma\beta^s n_s-2}\right)^2\right). \tag{A.23}$$

The first term in Eq. (A.23) is positive because

$$\frac{\partial\left(\beta\left(\frac{\gamma}{2}\beta+1\right)\right)}{\partial\beta} = \beta\gamma + 1 > 0$$

and from Lemma 14 we have $\beta^s > \beta^c$. Also,

$$\frac{1}{n_S+1} > \frac{1}{2-\beta^s\gamma n_S} > 0$$
$$\beta^s\gamma < -\frac{n_S-1}{n_S}$$

since $H\left(-\frac{n_S-1}{\gamma n_S}\right) = \frac{1}{n_S}\,(n_S-1)\,(n_D-2) > 0$ and the second term is positive, which implies $\lim_{\rho\to 1}\Delta V^i > 0$. □

Lemma 13. ΔV^i *is monotonically decreasing in* ρ*, i.e.,* $\frac{\partial \Delta V^i}{\partial \rho} < 0$ *for all* ρ.

Proof. From the definition of ΔV^i we have

$$\frac{\partial \Delta V^i}{\partial\rho} = \left(\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\left(\left(1-\pi_\eta^c\right)^2 + \left(\pi_\eta^c\right)^2(n_I-1)\right) - \beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\left(\left(1-\pi_\eta^s\right)^2 + \left(\pi_\eta^s\right)^2(n_S-1)\right)\right)\frac{1}{\rho^2}\sigma_\theta^2. \tag{A.24}$$

Since β^c, β^s, π_η^c and π_η^s are independent of ρ, monotonicity follows from Eq. (A.24). Note that

$$sign\left[\frac{\partial\Delta V^i}{\partial\rho}\right] = sign\left[\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\left(\left(1-\pi_\eta^c\right)^2 + \pi_\eta^{c2}(n_I-1)\right) - \beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\left(\left(1-\pi_\eta^s\right)^2 + \pi_\eta^{s2}(n_S-1)\right)\right] \tag{A.25}$$

where

$$\pi_\eta^s \equiv \frac{\beta^s}{n_S\beta^s + b^s} = \frac{1}{n_S}\frac{n_D-2\beta^s\gamma n_S}{2n_D - 2\beta^s\gamma n_S} \quad \text{and} \quad \pi_\eta^c \equiv \frac{1}{n_I+n_D}.$$

Rewriting the right hand side of Eq. (A.25) we have

$$sign\left[\frac{\partial\Delta V^i}{\partial\rho}\right] = sign\left[\begin{array}{c}\left(\beta^c\left(\frac{\gamma}{2}\beta^c+1\right) - \beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\right)\left(\left(1-\pi_\eta^c\right)^2 + \pi_\eta^{c2}(n_I-1)\right)\\ -\beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\left(\left(1-\pi_\eta^s\right)^2 + \pi_\eta^{s2}(n_S-1) - \left(1-\pi_\eta^c\right)^2 - \pi_\eta^{c2}(n_I-1)\right)\end{array}\right].$$

The first term is negative since $\frac{\partial\left(\beta\left(\frac{\gamma}{2}\beta+1\right)\right)}{\partial\beta} > 0$ and from Lemma 14 below we have $\beta^s > \beta^c$.

From Lemma 11 below we know that $\pi_\eta^s > \pi_\eta^c$. Let

$$\Omega\left(x\right) \equiv \left(1-x\right)^2 + x^2\left(n_S-1\right).$$

Then

$$\Omega'\left(x\right) = -2\left(1-x\right) + 2x\left(n_S-1\right) = 2\left(xn_S-1\right).$$

Since $\pi_\eta^s n_S < 1$ and $n_S\pi_\eta^c < 1$ we have that $\Omega'\left(x\right) < 0$ for all x in $\left[\pi_\eta^c, \pi_\eta^s\right]$. Since

$$\begin{aligned}
&\left(1-\pi_\eta^s\right)^2 + \pi_\eta^{s2}\left(n_S-1\right) - \left(1-\pi_\eta^c\right)^2 - \pi_\eta^{c2}\left(n_I-1\right) \\
&= \left(1-\pi_\eta^s\right)^2 + \pi_\eta^{s2}\left(n_S-1\right) - \left(\left(1-\pi_\eta^c\right)^2 + \pi_\eta^{c2}\left(n_S-1\right)\right) - \pi_\eta^{c2}\left(n_I-n_S\right) \\
&= \left(1-\pi_\eta^s\right)^2 + \pi_\eta^{s2}\left(n_S-1\right) - \left(\left(1-\pi_\eta^c\right)^2 + \pi_\eta^{c2}\left(n_S-1\right)\right) - \pi_\eta^{c2}n_S\left(n_D-1\right) \\
&= \Omega\left(\pi_\eta^s\right) - \Omega\left(\pi_\eta^c\right) - \pi_\eta^{c2}n_S\left(n_D-1\right) < 0
\end{aligned}$$

we have $\frac{\partial\Delta V^i}{\partial\rho} < 0$. □

Lemma 14. ***(Price sensitivities in fragmented and centralized markets)*** *Investors are more sensitive to the price in a centralized market than in a fragmented market structure, i.e.,* $\beta^c < \beta^\ell$*.*

Proof. Note that

$$\beta^c = -\frac{1}{\gamma}\frac{n_I+n_D-2}{n_I+n_D-1} < -\frac{1}{\gamma}\frac{n_I-1}{n_I}.$$

Since β^ℓ is given by $H\left(\beta^\ell\right) = 0$ where

$$H\left(\beta\right) = -2n\left(n-1\right)\left(\gamma\beta\right)^2 + \left(\left(2n-1\right)n_D - 2n\left(n-2\right)\right)\gamma\beta + 2\left(n-1\right)n_D.$$

and

$$H\left(-\frac{1}{\gamma}\frac{n_I-1}{n_I}\right) = -\frac{1}{n_S n_D^2}\left(2n_S^2 n_D\left(n_D-1\right) + n_S n_D^2\left(n_D-2\right) + 2\left(n_S-1\right) + n_D^2\right) < 0,$$

we have $\beta^s > -\frac{1}{\gamma}\frac{\left(n_I-1\right)}{n_I} > \beta^c$ because $\beta^c < 0, H'' < 0$ and $H\left(0\right) > 0$. □

Proof of Lemma 3

The expected utility of a dealer ℓ in a symmetric market structure is

$$W_D^\ell = -\mathbb{E}\left[\frac{\gamma}{2}\left(q_1^\ell + q_2^\ell\right)^2 + p_1^\ell q_1^\ell + p_2 q_2^\ell\right], \tag{A.26}$$

where q_1^ℓ and q_2^ℓ are the equilibrium quantities acquired by the dealer in the local and interdealer markets, respectively, and p_1 and p_2 are the equilibrium prices in these markets.

Using the coefficients for the equilibrium linear strategies in Eqs. (A.6), we have

$$\begin{aligned}
\mathbb{E}\left[\left(q_1^\ell + q_2^\ell\right)^2\right] &= \mathbb{E}\left[\left(\frac{2}{n_D}q_1^\ell + \frac{\left(n_D-2\right)}{n_D}\frac{\sum_{l\in N_D, l\neq\ell} q_1^l}{n_D-1}\right)^2\right] \\
&= \left(a^\ell + b^\ell\pi_\theta\right)^2\sigma_\theta^2 + \left(\frac{2}{n_D}b^\ell\pi_\eta\right)^2 n^\ell\sigma_\eta^2 + \left(\frac{\left(n_D-2\right)}{n_D}b^\ell\pi_\eta\right)^2\frac{n^\ell\sigma_\eta^2}{n_D-1} \\
&= \left(\frac{n^\ell\beta^\ell}{b^\ell + n^\ell\beta^\ell}\left(a^\ell + b^\ell\right)\right)^2\sigma_\theta^2 + \left(\frac{\beta^\ell}{n^\ell\beta^\ell + b^\ell}b^\ell\right)^2\frac{n^\ell\sigma_\eta^2}{n_D-1},
\end{aligned}$$

$$\mathbb{E}\left[p_1^\ell q_1^\ell\right] = \mathbb{E}\left[\left(\pi_\theta \theta + \pi_\eta \sum_{i\in N^\ell} \eta^i\right)\left(a^\ell \theta + b^\ell\left(\pi_\theta \theta + \pi_\eta \sum_{i\in N^\ell} \eta^i\right)\right)\right],$$
$$= -\frac{n^\ell \beta^\ell}{\left(b^\ell + n^\ell \beta^\ell\right)^2}\left(a^\ell + b^\ell\right)\left(n^\ell \alpha^\ell + a^\ell\right)\sigma_\theta^2 + b^\ell\left(\frac{\beta^\ell}{n^\ell \beta^\ell + b^\ell}\right)^2 n^\ell \sigma_\eta^2$$

and

$$\mathbb{E}\left[p_2 q_2^\ell\right] = \mathbb{E}\left[\left(\gamma \frac{\sum_{l\in N_D} q_1^l}{n_D}\right)\frac{(n_D-2)}{n_D}\left(\frac{\sum_{l\in N_D, l\neq \ell} q_1^l}{n_D - 1} - q_1^\ell\right)\right]$$
$$= \gamma \frac{(n_D-2)}{n_D}\left(b^\ell \pi_\eta\right)^2\left(\frac{1}{n_D} n^\ell \sigma_\eta^2 - \frac{1}{n_D} n^\ell \sigma_\eta^2\right) = 0.$$

Then, Eq. (A.26) becomes

$$V_D\left(m_n, \rho\right) = \left(\frac{n^\ell \beta^\ell}{b^\ell + n^\ell \beta^\ell}\right)^2\left(\left(-\frac{\gamma}{2}\left(a^\ell + b^\ell\right) + \frac{\left(n^\ell \alpha^\ell + a^\ell\right)}{n^\ell \beta^\ell}\right)\left(a^\ell + b^\ell\right) - \left(1 + \frac{\gamma}{2}\frac{1}{n_D - 1} b^\ell\right)\frac{b^\ell}{n^\ell}\frac{1-\rho}{\rho}\right)\sigma_\theta^2. \quad \text{(A.27)}$$

In a centralized market

$$V_D\left(m_{n_I}, \rho\right) = \frac{1}{2}\frac{1}{\gamma}\frac{n_I\left(n_I + n_D - 2\right)}{\left(n_I + n_D - 1\right)^2\left(n_I + n_D\right)}\left(n_I + \frac{1-\rho}{\rho}\right)\sigma_\theta^2 \quad \text{(A.28)}$$

In a fragmented symmetric market structure, using the expression for welfare in Eq. (A.27) we have

$$\frac{\partial V_D}{\partial \rho} = \left(\frac{n^\ell \beta^\ell}{b^\ell + n^\ell \beta^\ell}\right)^2\left(1 + \frac{\gamma}{2}\frac{1}{n_D - 1} b^\ell\right)\frac{b^\ell}{n^\ell}\frac{1}{\rho^2} < 0,$$

because

$$\left(1 + \frac{\gamma}{2}\frac{1}{n_D-1} b^\ell\right) = 1 - \frac{1}{2}\frac{1}{n_D - 1}\frac{n^\ell \beta^\ell n_D}{2n^\ell \beta^\ell - n_D}$$
$$= \frac{\left(2n_D(n_D-1) + (4-3n_D)n^\ell \beta^\ell\right)}{2(n_D-1)(n_D - 2n^\ell \beta^\ell)} > 0$$

since $n_D \geq 3$. In a centralized market structure, using the dealer's welfare in Eq. (A.28) we have

$$\frac{\partial V_D}{\partial \rho} = \left(\frac{n_I \beta^c}{\beta^c + n_I \beta^c}\right)^2\left(1 + \frac{\gamma}{2}\beta^c\right)\frac{\beta^c}{n_I}\frac{1}{\rho^2}\sigma_\theta^2$$
$$= -\frac{1}{2}\frac{1}{\gamma}\frac{n_I - 1}{n_I + 1}\frac{1}{\rho^2}\frac{1}{n_I}\sigma_\theta^2 < 0.$$

Proof of Proposition 7

Let $\Delta V^D \equiv \mathbb{E}\left(V^{\ell,c}\right) - \mathbb{E}\left(V^\ell\right)$. Since β^s is independent of ρ, ΔV^D is a continuous monotone function of ρ. From Lemma 15 and Lemma 16 below we have

$$\lim_{\rho\to 1} \Delta V^D = \mathbb{E}\left(V^{\ell,c}\right) - \mathbb{E}\left(V^\ell\right) > 0 \text{ and } \lim_{\rho\to 0} \Delta V^D = \mathbb{E}\left(V^{\ell,c}\right) - \mathbb{E}\left(V^\ell\right) < 0$$

which gives the result.

Lemma 15. *When investors have common priors we have*

$$\lim_{\rho\to 1} \Delta V^D > 0.$$

Proof. When investors have common priors and $\rho \to 1$ we have

$$\lim_{\rho\to 1} \Delta V^D = \frac{1}{2\gamma}\left(\frac{n_I^2\left(n_I + n_D - 2\right)}{\left(n_I + n_D - 1\right)^2\left(n_I + n_D\right)} - \frac{n_S \gamma \beta^s}{\beta^s \gamma n_S - 2}\right)\sigma_\theta^2 > 0$$

because

$$\frac{n_I^2 (n_I + n_D - 2)}{(n_I + n_D - 1)^2 (n_I + n_D)} > \frac{n_S \gamma \beta^s}{\beta^s \gamma n_S - 2}.$$

To see this note that

$$\frac{\partial \left(\frac{n_S x}{x n_S - 2} \right)}{\partial x} = -2 \frac{n_S}{(x n_S - 2)^2} < 0$$

and $-\frac{(n_S-1)}{n_S} > \gamma \beta^s$ since $H\left(-\frac{1}{\gamma}\frac{(n_S-1)}{n_S}\right) = \frac{1}{n_S}(n_S - 1)(n_D - 2) > 0$. Then,

$$-\frac{n_S \gamma \frac{1}{\gamma}\frac{(n_S-1)}{n_S}}{-\frac{1}{\gamma}\frac{(n_S-1)}{n_S}\gamma n_S - 2} > \frac{n_S \gamma \beta^s}{\beta^s \gamma n_S - 2}$$

and, because

$$\frac{n_I^2 (n_I + n_D - 2)}{(n_I + n_D - 1)^2 (n_I + n_D)} > -\frac{n_S \gamma \frac{1}{\gamma}\frac{(n_S-1)}{n_S}}{-\frac{1}{\gamma}\frac{(n_S-1)}{n_S}\gamma n_S - 2} = \frac{n_S - 1}{n_S + 1}(n_D - 1)(n_S + n_D + n_S n_D - 1) > 0$$

it follows that $\lim_{\rho \to 1} \Delta V^D > 0$. □

Lemma 16. *Dealers are better off in fragmented markets as the correlation in investor valuations disappears, i.e.*

$$\lim_{\rho \to 0} \Delta V^D < 0.$$

Proof. Note that

$$sign\left(\lim_{\rho \to 0} \Delta V^D\right) = sign\left(\frac{n_I (n_I + n_D - 2)}{(n_I + n_D - 1)^2 (n_I + n_D)} - \frac{1}{2}\gamma \beta^s \frac{\left(\left(2(n_D - 1) - \frac{1}{2}n_D\right) n_S \gamma \beta^s - n_D (n_D - 1)\right)}{(n_D - \beta^s \gamma n_S)^2}\frac{n_D}{n_D - 1}\right)$$

Let

$$R(\gamma \beta^s, n_D, n_S) = \frac{1}{2}\gamma \beta^s \frac{\left(\left(2(n_D - 1) - \frac{1}{2}n_D\right) n_S \gamma \beta^s - n_D (n_D - 1)\right)}{(n_D - \beta^s \gamma n_S)^2}\frac{n_D}{n_D - 1}$$

where

$$\frac{\partial R}{\partial x}(x, n_D, n_S) = \frac{1}{2}\frac{n_D^2}{(n_D - x n_S)^3 (n_D - 1)}((2n_D - 3) n_S x - n_D (n_D - 1)) < 0 \text{ for all } x < 0.$$

Then, since $-\frac{(n_S-1)}{n_S} > \gamma \beta^s$

$$R(\gamma \beta^s, n_D, n_S) > R\left(-\frac{(n_S - 1)}{n_S}, n_D, n_S\right) = \frac{1}{n_S}(n_S - 1)\frac{n_D (n_D - 1) + (n_S - 1)\left(\frac{3}{2}n_D - 2\right)}{(n_S + n_D - 1)^2}$$

Also,

$$\frac{\partial R}{\partial n_D}\left(-\frac{(n_S - 1)}{n_S}, n_D, n_S\right) = \frac{1}{2}\frac{(n_S - 1)(n_S + 1)}{n_S}\frac{3n_S + n_D - 3}{(n_S + n_D - 1)^3} > 0.$$

Then,

$$R\left(-\frac{(n_S - 1)}{n_S}, n_D, n_S\right) > R\left(-\frac{(n_S - 1)}{n_S}, 3, n_S\right) = \frac{4}{3}\frac{n_S}{n_S + 1}\frac{3n_S + 1}{(3n_S + 2)^2}.$$

Let

$$R(n_D, n_S) := \frac{n_S ((n_S + 1) n_D - 2)}{((n_S + 1) n_D - 1)^2 (n_S + 1)}$$

where

$$\frac{\partial L}{\partial n_D}(n_D, n_S) = -\frac{n_S}{(n_D + n_S n_D - 1)^3}(n_D + n_S n_D - 3) < 0.$$

Then,

$$L(n_D, n_S) < L(3, n_S) = \frac{4}{3}\frac{n_S}{n_S+1}\frac{3n_S+1}{(3n_S+2)^2}.$$

Since, as shown in Lemma 17 below

$$L(3, n_S) < R\left(-\frac{(n_S-1)}{n_S}, 3, n_S\right)$$

we have

$$L(n_D, n_S) < L(3, n_S) < R\left(-\frac{(n_S-1)}{n_S}, 3, n_S\right) < R\left(-\frac{(n_S-1)}{n_S}, n_D, n_S\right) < R\left(\gamma\beta^\ell, n_D, n_S\right)$$

for all $n_D \geq 3$, which implies

$$\lim_{\rho\to 0} \Delta V^D < 0.$$

□

Lemma 17.

$$L(3, n_S) < R\left(-\frac{(n_S-1)}{n_S}, 3, n_S\right)$$

Proof. We have

$$L(3, n_S) - R\left(-\frac{(n_S-1)}{n_S}, 3, n_S\right) = -\frac{1}{6}\frac{\left(111n_S^3 - 317\right)n_S^2 + \left(265n_S^3 - 312\right)n_S + \left(49n_S^3 - 84\right)}{n_S(n_S+1)(n_S+2)^2(3n_S+2)^2} < 0,$$

for all $n_S \geq 3$. □

Proof of Lemma 4

Volume in the interdealer market in a fragmented symmetric market structure is

$$\mathcal{V}_D = \frac{1}{2}\mathbb{E}\left[\sum_{\ell=1}^{n_D}\left|q_2^\ell\right|\right] = \frac{1}{2}\mathbb{E}\left[\sum_{\ell=1}^{n_D}\left|\frac{(n_D-2)}{(n_D-1)}\left(\frac{\sum_{l\in N_D} q_1^l}{n_D} - q_1^\ell\right)\right|\right],$$

where

$$q_2^\ell \sim N\left(0, \sigma^2_{q_2^\ell}\right)$$

with

$$\sigma^2_{q_2^\ell} = \frac{(n_D-2)^2}{(n_D-1)n_D}\left(\frac{b^\ell\beta^\ell}{n_S\beta^\ell + b^\ell}\right)^2 n_S\frac{1-\rho}{\rho}\sigma^2_\theta,$$

since

$$q_2^\ell = \frac{(n_D-2)}{(n_D-1)}\left(\frac{\sum_{l\in N_D} q_1^l}{n_D} - q_1^\ell\right) = \frac{(n_D-2)}{(n_D-1)}b^\ell\left(\frac{\sum_{l\in N_D}\left(p_1^l - p_1^\ell\right)}{n_D}\right)$$
$$= \frac{(n_D-2)}{(n_D-1)}\frac{b^\ell\beta^\ell}{n_S\beta^\ell + b^\ell}\left(\frac{\sum_{l\in N_D, l\neq\ell}\left(\sum_{i\in N_I(l)}\eta^i - \sum_{i\in N_I(\ell)}\eta^i\right)}{n_D}\right).$$

$\sigma^2_{q_2^\ell}$ is decreasing in ρ with $\lim_{\rho\to 1}\sigma^2_{q_2^\ell} = 0$. Since

$$\mathcal{V}_D = \sqrt{\frac{1}{2\pi}}\sigma_{q_2^\ell} n_D \tag{A.29}$$

we have $\frac{\partial \mathcal{V}_D}{\partial\rho} < 0$ and $\lim_{\rho\to 1}\mathcal{V}_D = 0$.

Proof of Lemma 5

The volume traded in the local markets in a fragmented market structure is $n_D \mathcal{V}^\ell$ where

$$\mathcal{V}^\ell = \frac{1}{2}\mathbb{E}\left[\frac{\sum_{i\in N_I(\ell)} \left|x_{1i}^\ell\right| + \left|q_1^\ell\right|}{n^\ell + 1}\right].$$

We know that

$$x_1^\ell \sim N\left(0, \sigma^2_{x_1^\ell}\right) \text{ and } q_1^\ell \sim N\left(0, \sigma^2_{q_1^\ell}\right),$$

where

$$\begin{aligned}
\sigma^2_{x_1} &= \mathbb{V}ar\left(x_1^\ell\right) = \mathbb{V}ar\left(-\beta^\ell\left(\theta^i - p_1^\ell\right)\right) \\
&= \left(\beta^\ell\right)^2\left(\mathbb{V}ar\left(\theta^i\right) + \mathbb{V}ar\left(p_1^\ell\right) - 2\mathbb{C}ov\left(\theta^i, p_1^\ell\right)\right) \\
&= \frac{\left(\beta^\ell\right)^2}{\left(b^\ell + n^\ell\beta^\ell\right)^2}\left(\left(a^\ell + b^\ell\right)^2 + \left(\left(\left(n^\ell - 1\right)\beta^\ell + b^\ell\right)^2 + \left(n^\ell - 1\right)\left(\beta^\ell\right)^2\right)\frac{1-\rho}{\rho}\right)\sigma^2_\theta,
\end{aligned}$$

and

$$\begin{aligned}
\sigma^2_{q_1} &= \mathbb{V}ar\left(q_1^\ell\right) = \mathbb{V}ar\left(a^\ell\theta + b^\ell p_1^\ell\right) \\
&= \left(a^\ell\right)^2\mathbb{V}ar\left(\theta\right) + \left(b^\ell\right)^2\mathbb{V}ar\left(p_1^\ell\right) + 2a^\ell b^\ell\mathbb{C}ov\left(\theta, p_1^\ell\right) \\
&= \frac{n^\ell\left(\beta^\ell\right)^2}{\left(b^\ell + n^\ell\beta^\ell\right)^2}\left(n^\ell\left(a^\ell + b^\ell\right)^2 + \left(b^\ell\right)^2\frac{1-\rho}{\rho}\right)\sigma^2_\theta.
\end{aligned}$$

Then, since for $x \sim N\left(0, \sigma^2\right)$, $|x|$ is a folded normal with $\mathbb{E}\left[|x|\right] = \sqrt{\frac{2}{\pi}}\,\sigma$, we have

$$\mathcal{V}^\ell = \frac{1}{\sqrt{2\pi}}\left(\frac{n^\ell\sigma_{x_1} + \sigma_{q_1^\ell}}{n^\ell + 1}\right). \tag{A.30}$$

In a centralized market, volume is given by

$$\mathcal{V}_c = \frac{1}{2}\mathbb{E}\left[\frac{\sum_{i=1}^{n_I}\left|x_{1i}^c\right| + n_D\left|q_1^c\right|}{n_I + n_D}\right].$$

We know that

$$x_{1i}^c \sim N\left(0, \sigma^2_{x_1^c}\right) \text{ and } q_1^c \sim N\left(0, \sigma^2_{q_1^c}\right),$$

where

$$\begin{aligned}
\sigma^2_{x_1^c} &= \mathbb{V}ar\left(x_1^c\right) = \mathbb{V}ar\left(-\beta^c\left(\theta^i - p_1^c\right)\right) \\
&= \left(\beta^c\right)^2\left(\mathbb{V}ar\left(\theta^i\right) + \mathbb{V}ar\left(p_1^\ell\right) - 2\mathbb{C}ov\left(\theta^i, p_1^\ell\right)\right) \\
&= \left(\beta^c\right)^2\frac{\frac{\left(n_I + n_D\right)^2}{\rho} + \left(n_I + \frac{1-\rho}{\rho}\right)\left(2n_D - n_I\right)}{\left(n_I + n_D\right)^2}\sigma^2_\theta
\end{aligned}$$

and

$$\begin{aligned}
\sigma^2_{q_1^c} &= \mathbb{V}ar\left(q_1^\ell\right) = \mathbb{V}ar\left(\beta^c p_1^\ell\right) \\
&= \left(\beta^c\right)^2\mathbb{V}ar\left(p_1^\ell\right) = \left(\beta^c\right)^2\frac{n_I^2 + n_I\frac{1-\rho}{\rho}}{\left(n_I + n_D\right)^2}\sigma^2_\theta.
\end{aligned}$$

Then,

$$\mathcal{V}_c = \frac{1}{\sqrt{2\pi}}\left(\frac{n_I\sigma_{x_1^c} + n_D\sigma_{q_1^c}}{n_I + n_D}\right) \tag{A.31}$$

Because a^ℓ, β^ℓ, and b^ℓ do not depend on ρ, a) and b) follow directly from Eq. (A.30) and Eq. (A.29), respectively since $\sigma_{x_1^i}$, $\sigma_{q_1^\ell}$, and $\sigma_{q_2^\ell}$ are decreasing in ρ. The third claim c) follows from Eq. (A.31) since $\sigma_{x_1^c}$ and $\sigma_{q_1^c}$ are decreasing in ρ because β^c does not depend on ρ.

B Fragmentation and disagreement data

We use three databases to explore the relation between disagreement and market fragmentation. We get analysts' price forecasts from IBES price target data, the total number of shares traded in exchanges from CRSP, and the total number of shares traded in ATS from FINRA ATS. After merging the three databases we are left with \$10,534\$ stock-month observations. The average number of monthly forecasts is close to 8.27. On average, disagreement is close to 11% of the average price.

We construct our measure of fragmentation similarly to O'Hara and Ye (2011). As a proxy for market fragmentation, we use the fraction of shares of a given stock that is traded in ATS out of the total shares of the stocks traded in a given month. The FINRA ATS data is reported at a weekly frequency. We aggregate all shares across all ATS for each week and then across all weeks in a month to get the monthly shares of each stock traded in ATS which we label $\#TotalShares_{ATS}^{XYZ}$ for stock XYZ. We obtain the number of shares traded in the main exchanges in a month from CRSP by aggregating the daily number of shares sold at the CUSIP level. We label this amount $\#TotalShares_{nonATS}^{XYS}$ for stock XYZ. Our measure of fragmentation is given by

$$Fragmentation = \frac{\#TotalShares_{ATS}^{XYZ}}{\#TotalShares_{ATS}^{XYZ} + \#TotalShares_{nonATS}^{XYS}}.$$

We measure disagreement as the standard deviation in analyst forecasts for a particular stock normalized by the average of the forecast prices to control for the differences in the scale of stock prices. We label this measure $DISP^{XYZ}$ for stock XYZ. We use the CUSIP identifier from CRSP to download the corresponding IBES price target data. The IBES price target data reports analysts' forecasts of each stock price. We focus on forecasts with 12 month horizons on their announcement dates. Using the IBES foreign exchange data we convert all forecasts to USD. Then, for each stock XYZ with more than three forecasts within a month, we compute our measure of disagreement by taking the standard deviation of those forecasts normalized by the average of the forecast prices.

C Online Appendix (not for publication)

This section contains intermediate results used in the main appendix.

Lemma 18. *In a symmetric equilibrium*

$$a^{sym} = -n^{\ell 2}\beta^{sym2}\frac{\gamma}{n^{\ell}\beta^{sym}\gamma - 2}\frac{n_D - 2}{n_D - 2n^{\ell}\beta^{sym}\gamma} \tag{C.1}$$

where

$$\lim_{n_D\to\infty} a^{sym} = \frac{\gamma\left(n^{\ell}\bar{\beta}\right)^2}{2-\gamma\bar{\beta}n^{\ell}}\ ,\quad \lim_{n^{\ell}\to\infty} a^{sym} = \lim_{n^{\ell}\to\infty} -b^{sym}$$

and

$$\lim_{n_D\to 3} a^{sym} = \frac{\frac{1}{3}n^{\ell}\left(b^{sym}\right)^2\gamma\beta^{sym}}{n^{\ell}\beta^{sym} + b^{sym} - \frac{1}{3}b^{sym}n^{\ell}}.$$

If one investor deviates from a fragmented symmetric market structure then

$$\lim_{n_D\to\infty}\begin{bmatrix} a^{\ell} \\ a^{h} \\ a^{o}\end{bmatrix} = \begin{bmatrix} -\gamma\frac{n^{\ell}\beta^{o}}{n^{\ell}\beta^{o}\gamma-2}\left(n^{\ell}-1\right)\bar{\beta}^{\ell} \\ -\gamma\frac{n^{\ell}\beta^{o}}{n^{\ell}\beta^{o}\gamma-2}\left(n^{\ell}+1\right)\bar{\beta}^{h} \\ -\gamma\frac{n^{\ell}\beta^{o}}{n^{\ell}\beta^{o}\gamma-2}n^{\ell}\beta^{o}\end{bmatrix}$$

and

$$\lim_{n_D\to 3}\begin{bmatrix} a^{\ell} \\ a^{h} \\ a^{o}\end{bmatrix} = \begin{bmatrix} b^{\ell}\gamma\left(5b^{\ell}\gamma+12\right)\frac{5b^{h+o}\gamma+6b^{h}+6b^{o}}{90b^{h+o}\gamma^2+288b^{h}\gamma+288b^{o}\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^{\ell}\gamma+864} \\ b^{h}\gamma\left(5b^{h}\gamma+12\right)\frac{6b^{\ell}+5b^{o+\ell}\gamma+6b^{o}}{90b^{h+o}\gamma^2+288b^{h}\gamma+288b^{o}\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^{\ell}\gamma+864} \\ b^{o}\gamma\left(5b^{o}\gamma+12\right)\frac{6b^{\ell}+5b^{h+\ell}\gamma+6b^{h}}{90b^{h+o}\gamma^2+288b^{h}\gamma+288b^{o}\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^{\ell}\gamma+864}\end{bmatrix}$$

Proof. In a symmetric equilibrium

$$a^{sym} = \frac{n_D-2}{n_D}b^{sym}n^{\ell}\frac{a^{sym}+b^{sym}}{n^{\ell}\beta^{sym}+b^{sym}}\gamma\beta^{sym}$$

or, alternatively,

$$a^{sym} = \frac{\frac{n_D-2}{n_D}n^{\ell}\left(b^{sym}\right)^2\gamma\beta^{sym}}{\left(n^{\ell}\beta^{sym}+b^{sym}-\frac{n_D-2}{n_D}b^{sym}n^{\ell}\gamma\beta^{sym}\right)}$$

where

$$b^{sym} = -\frac{n^{\ell}\beta^{sym}n_D}{2\gamma n^{\ell}\beta^{sym}-n_D}$$

and $\beta^{sym} = \beta\left(n^{\ell}\right)$. Then,

$$\begin{aligned}\lim_{n_D\to\infty} a^{sym} &= \lim_{n_D\to\infty}\frac{\frac{n_D-2}{n_D}n^{\ell}\left(b^{sym}\right)^2\gamma\beta^{sym}}{n^{\ell}\beta^{sym}+b^{sym}-\frac{n_D-2}{n_D}b^{sym}n^{\ell}\gamma\beta^{sym}} \\ &= \frac{\gamma\left(n^{\ell}\bar{\beta}\right)^2}{2-\gamma\bar{\beta}n^{\ell}} = \frac{2n^{2\ell}\left(n^{\ell}-1\right)^2}{\gamma\left(2n^{\ell}-1\right)\left(n^{\ell}+n^{2\ell}-1\right)}\end{aligned}$$

$$\lim_{n^{\ell}\to\infty} a^{sym} = \lim_{n^{\ell}\to\infty} -b^{sym} = \frac{n_D}{2\gamma}$$

$$\lim_{n^{\ell}\to 3} a^{sym} = \frac{\frac{1}{3}\left(b^{sym}\right)^2\gamma n^{\ell}\beta^{sym}}{n^{\ell}\beta^{sym}+b^{sym}-\frac{1}{3}b^{sym}n^{\ell}}$$

Using that

$$n^{\ell}\beta^{sym}=\frac{3b^{sym}}{2\gamma b^{sym}+3}$$

$$\lim_{n^{\ell}\to 3}a^{sym}=\frac{\left(b^{sym}\right)^{2}\gamma}{3+\left(1-\frac{1}{3}n^{\ell}\right)\left(2\gamma b^{sym}+3\right)}=\frac{\left(b^{sym}\right)^{2}\gamma}{6+\left(1-\frac{1}{3}n^{\ell}\right)2\gamma b^{sym}-n^{\ell}}$$

If the investor chooses to deviate, the market structure is as follows: market ℓ has $n^{\ell}-1$ investors, market h has $n^{\ell}+1$ and the rest of the n_D-2 markets have n^{ℓ} investors. Then,

$$a^{\ell}=b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\sum_{l\in N_D,l\neq\ell}n^{l}\frac{a^{l}\beta^{l}-b^{l}\alpha^{l}}{n^{l}\beta^{l}+b^{l}}$$

$$a^{\ell}=b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-2)\,n^{\ell}\frac{a^{o}+b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o}+\left(n^{\ell}+1\right)\frac{a^{h}+b^{h}}{\left(n^{\ell}+1\right)\beta^{h}+b^{h}}\beta^{h}\right)$$
$$a^{h}=b^{h}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-2)\,n^{\ell}\frac{a^{o}+b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o}+\left(n^{\ell}-1\right)\frac{a^{\ell}+b^{\ell}}{\left(n^{\ell}-1\right)\beta^{\ell}+b^{\ell}}\beta^{\ell}\right)$$
$$a^{o}=b^{o}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-3)\,n^{\ell}\frac{a^{o}+b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o}+\left(n^{\ell}-1\right)\frac{a^{\ell}+b^{\ell}}{\left(n^{\ell}-1\right)\beta^{\ell}+b^{\ell}}\beta^{\ell}+\left(n^{\ell}+1\right)\frac{a^{h}+b^{h}}{\left(n^{\ell}+1\right)\beta^{h}+b^{h}}\beta^{h}\right)$$

One can rewrite this system as

$$\begin{bmatrix}a^{\ell}\\a^{h}\\a^{o}\end{bmatrix}=\left[I_3-\begin{bmatrix}\Omega_{11}&\Omega_{12}&\Omega_{13}\\\Omega_{21}&\Omega_{22}&\Omega_{23}\\\Omega_{31}&\Omega_{32}&\Omega_{33}\end{bmatrix}\right]^{-1}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}\tag{C.2}$$

where

$$\begin{bmatrix}\Omega_{11}&\Omega_{12}&\Omega_{13}\\\Omega_{21}&\Omega_{22}&\Omega_{23}\\\Omega_{31}&\Omega_{32}&\Omega_{33}\end{bmatrix}=\begin{bmatrix}0&b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{\left(n^{\ell}+1\right)\beta^{h}}{(n^{\ell}+1)\beta^{h}+b^{h}}&b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{(n_D-2)}{n_D-1}\frac{n^{\ell}\beta^{o}}{n^{\ell}\beta^{o}+b^{o}}\\b^{h}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{\left(n^{\ell}-1\right)\beta^{\ell}}{(n^{\ell}-1)\beta^{\ell}+b^{\ell}}&0&b^{h}\gamma\frac{(n_D-2)}{n_D}\frac{(n_D-2)}{n_D-1}\frac{n^{\ell}\beta^{o}}{n^{\ell}\beta^{o}+b^{o}}\\b^{o}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{\left(n^{\ell}-1\right)\beta^{\ell}}{(n^{\ell}-1)\beta^{\ell}+b^{\ell}}&b^{o}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{\left(n^{\ell}+1\right)\beta^{h}}{(n^{\ell}+1)\beta^{h}+b^{h}}&b^{o}\gamma\frac{(n_D-2)}{n_D}\frac{(n_D-3)}{n_D-1}\frac{n^{\ell}\beta^{o}}{n^{\ell}\beta^{o}+b^{o}}\end{bmatrix}$$

and

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-2)\frac{n^{\ell}b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o}+\frac{\left(n^{\ell}+1\right)b^{h}}{(n^{\ell}+1)\beta^{h}+b^{h}}\beta^{h}\right)\\b^{h}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-2)\frac{n^{\ell}b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o}+\frac{\left(n^{\ell}-1\right)b^{\ell}}{(n^{\ell}-1)\beta^{\ell}+b^{\ell}}\beta^{\ell}\right)\\b^{o}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-3)\frac{n^{\ell}b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o}+\frac{\left(n^{\ell}-1\right)b^{\ell}}{(n^{\ell}-1)\beta^{\ell}+b^{\ell}}\beta^{\ell}+\frac{\left(n^{\ell}+1\right)b^{h}}{(n^{\ell}+1)\beta^{h}+b^{h}}\beta^{h}\right)\end{bmatrix}.$$

Then,

$$\lim_{n_D\to\infty}\begin{bmatrix}a^{\ell}\\a^{h}\\a^{o}\end{bmatrix}=\left[I_3-\lim_{n_D\to\infty}\begin{bmatrix}\Omega_{11}&\Omega_{12}&\Omega_{13}\\\Omega_{21}&\Omega_{22}&\Omega_{23}\\\Omega_{31}&\Omega_{32}&\Omega_{33}\end{bmatrix}\right]^{-1}\lim_{n_D\to\infty}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-n^{\ell}\beta^{o+\ell}\gamma\frac{n^{\ell}-1}{n^{\ell}\beta^{o}\gamma-2}\\-n^{\ell}\beta^{h+o}\gamma\frac{n^{\ell}+1}{n^{\ell}\beta^{o}\gamma-2}\\-n^{2\ell}\beta^{2o}\frac{\gamma}{n^{\ell}\beta^{o}\gamma-2}\end{bmatrix}$$

$$\lim_{n_D\to\infty}\begin{bmatrix}a^{\ell}\\a^{h}\\a^{o}\end{bmatrix}=\left[I_3-\lim_{n_D\to\infty}\begin{bmatrix}\Omega_{11}&\Omega_{12}&\Omega_{13}\\\Omega_{21}&\Omega_{22}&\Omega_{23}\\\Omega_{31}&\Omega_{32}&\Omega_{33}\end{bmatrix}\right]^{-1}\lim_{n_D\to\infty}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}\frac{2}{\gamma}\frac{\left(n^{\ell}-1\right)^{2}\left(n^{\ell}-2\right)}{(2n^{\ell}-3)(n^{\ell2}+n^{\ell}-1)}\\\frac{2}{\gamma}\frac{n^{\ell}\left(n^{\ell}-1\right)\left(n^{\ell}+1\right)}{(2n^{\ell}+1)(n^{\ell2}+n^{\ell}-1)}\\\frac{2}{\gamma}\frac{n^{\ell}\left(n^{\ell}-1\right)^{2}}{(2n^{\ell}-1)(n^{\ell2}+n^{\ell}-1)}\end{bmatrix}$$

When $n_D \to 3$, we have

$$\lim_{n_D\to 3}\begin{bmatrix} a^\ell \\ a^h \\ a^o \end{bmatrix} = \left[I_3 - \lim_{n_D\to 3}\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}\right]^{-1} \lim_{n_D\to 3}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} b^\ell\gamma\left(5b^\ell\gamma+12\right)\frac{5b^{h+o}\gamma+6b^h+6b^o}{90b^{h+o}\gamma^2+288b^h\gamma+288b^o\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^\ell\gamma+864} \\ b^h\gamma\left(5b^h\gamma+12\right)\frac{6b^\ell+5b^{o+\ell}\gamma+6b^o}{90b^{h+o}\gamma^2+288b^h\gamma+288b^o\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^\ell\gamma+864} \\ b^o\gamma\left(5b^o\gamma+12\right)\frac{6b^\ell+5b^{h+\ell}\gamma+6b^h}{90b^{h+o}\gamma^2+288b^h\gamma+288b^o\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^\ell\gamma+864} \end{bmatrix},$$

where we used that

$$n\beta(n) = \frac{3b(n)}{2\gamma b(n)+3}.$$

□

Lemma 19. *Let*

$$F(n) \equiv -\frac{(4\gamma b(n)n + 6n + 3\gamma b(n))}{n(\gamma b(n)+6)^2} b(n).$$

Then, $F(n) > 0$ for all $n \geq 0$ and $F'(n) < 0$.

Proof. First, note that

$$(4n+3)\gamma b(n) + 6n > 0 \tag{C.3}$$

since using that $b(n) = \frac{3n\beta(n)}{3-2n\gamma\beta(n)}$ Eq. (C.3) becomes

$$(4n+3)\gamma 3n\beta(n) + 6n(3-2n\gamma\beta(n)) = 9n(\gamma\beta(n)+2) > 0$$

since $\beta(n) \geq -\frac{1}{\gamma}$. The derivative of $F(\cdot)$ with respect to n is

$$F'(n) = \frac{3}{n(b(n)\gamma+6)^2}\left(\gamma\frac{b(n)^2}{n} - \frac{2}{(b(n)\gamma+6)}(6n+6b(n)\gamma+7b(n)n\gamma)\frac{\partial b(n)}{\partial n}\right)$$

Then, using that $b(n) = \frac{3n\beta(n)}{3-2n\gamma\beta(n)}$ we have

$$\frac{\partial b(n)}{\partial n} = \frac{9}{(2n\gamma\beta(n)-3)^2}\left(\beta(n) + 9n\frac{\partial\beta(n)}{\partial n}\right)$$

and

$$F'(n) = \frac{n}{3n^2(n\gamma\beta(n)-2)^3}\left((\beta(n)\gamma+2)(n\beta(n)\gamma+2)\beta(n) + 18n((n+2)\gamma\beta(n)+2)\frac{\partial\beta(n)}{\partial n}\right).$$

From Lemma 6 we know $\beta(n) \geq -\frac{1}{\gamma}$ and thus $(\beta(n)\gamma+2) > 0$. Moreover, $n\beta(n)\gamma + 2 < 0$ since

$$H\left(-\frac{2}{n\gamma}; n_D = 3\right) = \frac{2}{n}\left(5n^2 - 17n + 7\right) > 0 \text{ for } n \geq 3.$$

Then, $(n+2)\gamma\beta(n) + 2 < 0$ and $F'(n) < 0$. □

Lemma 20. *We show that*

$$\frac{n^\ell+1}{n^\ell} > \left(\frac{\gamma b^{dev}+6}{2\gamma b^{dev}+6}\left(1+\frac{a^{dev}}{b^{dev}}\right)\right)^2. \tag{C.4}$$

Proof. Using Lemma 18 we have

$$\frac{\left(\gamma b^{dev}+6\right)}{\left(2\gamma b^{dev}+6\right)}\left(1+\frac{a^{dev}}{b^{dev}}\right) = \frac{\left(\gamma b^{dev}+6\right)(5\gamma b^{sym}+12)\left(5\gamma b^\ell+12\right)}{288\gamma b^{sym}+288\gamma b^{dev}+288\gamma b^\ell+90\gamma^2 b^{sym}b^{dev}+90\gamma^2 b^{sym}b^\ell+90\gamma^2 b^{dev}b^\ell+25\gamma^3 b^{sym}b^{dev}b^\ell+864}.$$

Then, showing Eq. (C.4) holds is the same as showing that

$$\left(\frac{288\gamma b^{sym}+288\gamma b^{dev}+288\gamma b^{\ell}+90\gamma^2 b^{sym}b^{dev}+90\gamma^2 b^{sym}b^{\ell}+90\gamma^2 b^{dev}b^{\ell}+25\gamma^3 b^{sym}b^{dev}b^{\ell}+864}{(\gamma b^{dev}+6)(5\gamma b^{sym}+12)(5\gamma b^{\ell}+12)}\right)^2 > \frac{n^{\ell}}{n^{\ell}+1}$$

$$\left(-6\gamma\frac{\left(b^{\ell}-b^{dev}\right)}{\left(\gamma b^{dev}+6\right)\left(5\gamma b^{\ell}+12\right)} - 6\gamma\frac{\left(b^{sym}-b^{dev}\right)}{\left(5b^{sym}\gamma+12\right)\left(\gamma b^{dev}+6\right)}+1\right)^2 > \frac{n^{\ell}}{n^{\ell}+1}$$

$$\left(-G\left(b^{\ell};b^{dev}\right)-G\left(b^{sym};b^{dev}\right)+1\right)^2 > \frac{n^{\ell}}{n^{\ell}+1}, \qquad \text{(C.5)}$$

where

$$G\left(x;b^{dev}\right) = 6\gamma\frac{\left(x-b^{dev}\right)}{\left(\gamma b^{dev}+6\right)\left(5\gamma x+12\right)}.$$

Because

$$b\left(n\right) = \frac{3n\beta\left(n\right)}{\left(3-2\gamma n\beta\left(n\right)\right)}, \qquad \text{(C.6)}$$

we have

$$\frac{5\gamma b\left(n\right)+12}{\gamma b\left(n\right)+6} = \frac{\beta\left(n\right)\gamma\left(n+1\right)-4}{\beta\left(n\right)\gamma\left(n+1\right)-2} > 0,$$

which implies

$$G'\left(x\right) = \frac{5\gamma b^{dev}+12}{\left(\gamma b^{dev}+6\right)\left(5x\gamma+12\right)^2} > 0$$

and $G\left(x,b^{dev}\right)>0$ for $x>b^{dev}$ since $G\left(b^{dev};b^{dev}\right)=0$. Then, since $G\left(b^{\ell};b^{dev}\right)>G\left(b^{sym};b^{dev}\right)$, if

$$2G\left(b^{\ell};b^{dev}\right) = 12\gamma\frac{b^{\ell}-b^{dev}}{\left(5b^{\ell}\gamma+12\right)\left(\gamma b^{dev}+6\right)} < 1-\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}, \qquad \text{(C.7)}$$

Eq. (C.5) holds. Using that $\gamma\beta^{dev}\in(-1,0)$ and Eq. (C.6) we have

$$b\left(n\right) > -\frac{1}{\gamma}\frac{3n}{\left(3+2n\right)}.$$

Because the left hand side of Eq. (C.7) is decreasing in b^{dev} we can rewrite Eq. (C.7) as

$$4\frac{\left(2n+5\right)\gamma b^{\ell}+3\left(n+1\right)}{3\left(5b^{\ell}\gamma+12\right)\left(n+3\right)} < 1-\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}$$

$$\gamma b^{\ell} < -\frac{12\left(n^{\ell}+1+3\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)\right)}{4\left(2n^{\ell}+5\right)+15\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)}$$

since $4\left(2n^{\ell}+5\right)+15\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)>0$. Using Eq. (C.6) this becomes

$$\frac{3\left(n^{\ell}-1\right)\gamma\beta^{\ell}}{\left(3-2\gamma\left(n^{\ell}-1\right)\beta^{\ell}\right)} < -\frac{12\left(n^{\ell}+1+3\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)\right)}{4\left(2n^{\ell}+5\right)+15\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)}$$

$$\gamma\beta^{\ell} < \frac{4\left(-2\left(n^{\ell}+4\right)+3\left(3+n^{\ell}\right)\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}\right)}{\left(n^{\ell}-1\right)\left(-3n^{\ell}-13+3\left(3+n^{\ell}\right)\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}\right)} \equiv Z$$

which holds because

$$H\left(\frac{Z}{\gamma};n=n^{\ell}-1,n_D=3\right) = -2\left(n^{\ell}-1\right)\left(n^{\ell}-2\right)Z^2+\left(3\left(2n^{\ell}-3\right)+2\left(n^{\ell}-1\right)-2\left(n^{\ell}-1\right)\left(n^{\ell}-2\right)\right)Z+6\left(n^{\ell}-2\right) > 0$$

for all $n^\ell > 2$. Indeed, it is simple to check that

$$\left(n^\ell+3\right)\left(n^\ell+1\right)\left(-56n^\ell-8\left(n^\ell\right)^2+\left(n^\ell\right)^3-19\right)>0,$$

for $n^\ell \geq 3$. This implies that

$$6\sqrt{\frac{n^\ell}{n^\ell+1}}\left(n^\ell+3\right)\left(n^\ell+1\right)\left(-56n^\ell-8\left(n^\ell\right)^2+\left(n^\ell\right)^3-19\right)>6\frac{n^\ell}{n^\ell+1}\left(n^\ell+3\right)\left(n^\ell+1\right)\left(-56n^\ell-8\left(n^\ell\right)^2+\left(n^\ell\right)^3-19\right),$$

and further

$$\begin{aligned}949n^\ell+1313\left(n^\ell\right)^2+489\left(n^\ell\right)^3+33\left(n^\ell\right)^4-6\left(n^\ell\right)^5-58+6\sqrt{\frac{n^\ell}{n^\ell+1}}\left(n^\ell+3\right)\left(n^\ell+1\right)\left(-56n^\ell-8\left(n^\ell\right)^2+\left(n^\ell\right)^3-19\right)\\ >3\left(n^\ell\right)^4+9\left(n^\ell\right)^3+191\left(n^\ell\right)^2+607n^\ell-58,\end{aligned}$$

for $n^\ell \geq 13$.

This shows that $H\left(\frac{Z}{\gamma}; n=n^\ell-1, n_D=3\right) > 0$ for $n^\ell \geq 13$. For $3 \leq n^\ell \leq 12$ we show that $H\left(\frac{Z}{\gamma}; n=n^\ell-1, n_D=3\right) > 0$ point by point. □

Lemma 21. *We show that*

$$\frac{d\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\right)}{dn}=\frac{\partial\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\right)}{\partial\beta}\frac{\partial\beta}{\partial n}>0.$$

Proof.

$$\frac{\partial\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\right)}{\partial\beta\left(n\right)}=-\beta\left(n\right)\gamma-1<0$$

since

$$\begin{aligned}H\left(-\frac{1}{\gamma}\right)&=-2\left(n-1\right)n-\left(\left(2n-1\right)n_D-\left(n-2\right)2n\right)+2\left(n-1\right)n_D\\&=-2n-n_D<0,\end{aligned}$$

$$\beta\left(n\right)>-\frac{1}{\gamma}$$

□

Lemma 22. *We show that*

$$n_D+2\gamma n^2\frac{\partial\beta\left(n\right)}{\partial n}>0.$$

Proof. Using the definition of $\frac{\partial\beta(n)}{\partial n}$

$$n_D\left(-4\left(n-1\right)n\gamma\beta\left(n\right)+\left(\left(2n_I-1\right)n_D-\left(n-2\right)2n\right)\right)-2n^2\left(2\left(-\gamma^2\beta\left(n\right)^2\left(2n-1\right)-\gamma\beta\left(n\right)\left(2\left(n-1\right)-n_D\right)+n_D\right)\right)<0$$

$$4n^2\left(2n-1\right)\gamma^2\beta\left(n\right)^2+\left(-4n^2\left(n_D-2n+2\right)-4nn_D\left(n-1\right)\right)\gamma\beta\left(n\right)+\left(n_D\left(n_D\left(2n-1\right)-2n\left(n-2\right)\right)-4n^2n_D\right)<0$$

Using the definition of $\beta\left(n\right)^2$

$$\left(2n^2+\left(2n-1\right)n_D\right)\frac{\left(n-1\right)n_D+2\beta\left(n\right)\gamma n_I}{n-1}>0$$

since we show below in Lemma 24 that

$$\left(\left(n-1\right)n_D+2\gamma\beta\left(n\right)n\right)>0.$$

□

Lemma 23. *We show that* $\frac{\beta(n)b(n)}{(b(n)+n\beta(n))^2}$ *is decreasing in* n.

Proof.

$$\frac{\beta(n)\,b(n)}{(b(n)+n\beta(n))^2} = \frac{\beta(n)}{(b(n)+n\beta(n))}\frac{b(n)}{(b(n)+n\beta(n))}$$

Then,

$$\frac{d}{dn}\left(\frac{\beta(n)\,b(n)}{(b(n)+n\beta(n))^2}\right) = \frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right)\frac{b(n)}{b(n)+n\beta(n)} + \frac{\beta(n)}{b(n)+n\beta(n)}\frac{d}{dn}\left(\frac{b(n)}{(b(n)+n\beta(n))}\right)$$

From Lemma 25 and Lemma 24 we know that

$$\frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) < 0 \text{ and } \frac{d}{dn}\left(\frac{b(n)}{(b(n)+n\beta(n))}\right) < 0$$

Since $b(n) < 0$ and $\beta(n) < 0$ the result follows. □

Lemma 24. *We show that*

$$\frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) < 0.$$

Proof. We have that

$$\frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) = \frac{\partial}{\partial n}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) + \frac{\partial}{\partial b(n)}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right)\frac{db(n)}{dn} + \frac{\partial}{\partial\beta(n)}\left(\frac{\beta(n)}{(b(n)+n\beta(n))}\right)\frac{\partial\beta}{\partial n},$$

where

$$\frac{\partial}{\partial n}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) = -\frac{\beta(n)^2}{(b(n)+n\beta(n))^2} < 0$$
$$\frac{\partial}{\partial b(n)}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) = -\frac{\beta(n)}{(b(n)+n\beta(n))^2} > 0$$
$$\frac{\partial}{\partial\beta(n)}\left(\frac{\beta(n)}{(b(n)+n\beta(n))}\right) = \frac{b(n)}{(b(n)+n\beta(n))^2} < 0.$$

Then,

$$\frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) = \frac{\beta(n)^2}{(b(n)+n\beta(n))^2(n_D-2\gamma\beta(n)n)^2}\left(\left(-2\gamma n^2 n_D\right)\frac{\partial\beta(n)}{\partial n} - \left((n_D-2\gamma\beta(n)\,n)^2+n_D^2\right)\right),$$

where

$$2\gamma n^2 n_D\frac{\partial\beta(n)}{\partial n} + \left((n_D-2\gamma\beta(n)\,n)^2+n_D^2\right) > 0. \tag{C.8}$$

To see this note that using

$$\frac{\partial\beta(n)}{\partial n} = -\frac{2\left(-\beta(n)^2\gamma^2(2n-1)-\beta\gamma(2(n-1)-n_D)+n_D\right)}{-4(n-1)\,n\gamma^2\beta+((2n-1)\,n_D-(n-2)\,2n)\,\gamma}$$

Eq. (C.8) becomes

$$(n_D-2\gamma\beta(n)\,n)^2 + n^2 n_D\left(-\frac{2\left(-\gamma^2\beta(n)^2(2n-1)-\gamma\beta(n)(2(n-1)-n_D)+n_D\right)}{-4(n-1)\,n\gamma\beta(n)+((2n-1)\,n_D-(n-2)\,2n)}\right) + n_D^2 > 0$$

The first term is positive. The last two terms can be written as $n_D J\left(\beta\left(n\right)\right)$ where

$$J\left(\beta\left(n\right)\right) := 2n^2\left(-\frac{2\left(-\gamma^2\beta\left(n\right)^2\left(2n-1\right)-\gamma\beta\left(n\right)\left(2\left(n-1\right)-n_D\right)+n_D\right)}{-4\left(n-1\right)n\gamma\beta\left(n\right)+\left(\left(2n-1\right)n_D-\left(n-2\right)2n\right)}\right)+n_D>0$$

Rearranging terms

$$J\left(\beta\left(n\right)\right) = \frac{\left(\left(4\gamma^2n^2-8\gamma^2n^3\right)\beta^2+\left(8\gamma n^2-8\gamma n^3+8\gamma n^2n_D-4\gamma nn_D\right)\beta+(1-2n)n_D^2+\left(6n^2-4n\right)n_D\right)}{-(-4(n-1)n\gamma\beta+((2n-1)n_D-(n-2)2n))}$$

The denominator is negative. Substituting $\left(\beta\left(n\right)\right)^2$, the numerator can be written as

$$-\frac{1}{n-1}\left(\left(n-1\right)n_D+2\beta\left(n\right)\gamma n\right)\left(2n^2+\left(2n-1\right)n_D\right)<0$$

since

$$\begin{aligned}\left(\left(n-1\right)n_D+2\beta\left(n\right)\gamma n\right)&>0\\ \gamma\beta\left(n\right)&>-\frac{\left(n-1\right)n_D}{2n}\end{aligned}$$

because

$$H\left(-\frac{\left(n-1\right)n_D}{2n}\right)=-\frac{1}{2}n_In_D\left(n_I-1\right)\left(n_D-2\right)<0.$$

Then, $n_D J\left(\beta\left(n\right)\right)>0$ and Eq. (C.8) holds. □

Lemma 25. *We show that*

$$\frac{d}{dn}\left(\frac{b\left(n\right)}{b\left(n\right)+n\beta\left(n\right)}\right)<0.$$

Proof. Using the definition of $b\left(n\right)$ we have

$$\frac{b\left(n\right)}{b\left(n\right)+n\beta\left(n\right)}=\frac{\frac{nn_D\beta(n)}{(n_D-2n\gamma\beta(n))}}{\frac{nn_D\beta(n)}{(n_D-2\gamma n\beta(n))}+n\beta\left(n\right)}=\frac{n_D}{2\left(n_D-n\gamma\beta\left(n\right)\right)}$$

Then,

$$\begin{aligned}\frac{d}{dn}\left(\frac{b\left(n\right)}{b\left(n\right)+n\beta\left(n\right)}\right)&=\frac{d}{dn}\left(\frac{n_D}{2\left(n_D-n\gamma\beta\left(n\right)\right)}\right)=\frac{1}{2}\gamma\frac{n_D}{\left(n_D-n\gamma\beta\left(n\right)\right)^2}\left(\frac{d\left(n\beta\left(n\right)\right)}{dn}\right)\\&=\frac{1}{2}\gamma\frac{n_D}{\left(n_D-n\gamma\beta\left(n\right)\right)^2}\left(n\frac{\partial\beta\left(n\right)}{\partial n}+\beta\left(n\right)\right)<0.\end{aligned}$$

□