Collateralizing Liquidity

Cecilia Parlatore[*] [†]

New York University, Stern School of Business

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Abstract

I develop a dynamic model of optimal funding to understand why financial assets are used as collateral instead of being sold to raise funds. Firms need funds to invest in risky projects with non-observable returns. Since holding these assets allows firms to raise these funds, investing firms value the asset more than non-investing ones. When assets are less than perfectly liquid and investment opportunities are persistent, collateralized debt minimizes asset transfers from investing to non-investing firms and, thus, is optimal. Frictions in asset markets lead to an illiquidity discount and a collateral premium, which increase with the asset's illiquidity.

[*]Address: 44 W4th street, suite 9-87, New York, NY, 10012. e-mail: cparlato@stern.nyu.edu, Webpage: www.ceciliaparlatore.com.

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1 Introduction

Collateralized debt is a widely used form of financing. Trillions of dollars are traded daily in debt collateralized by diverse financial assets, such as sale and repurchase agreements (repos) and collateralized over-the-counter derivative trades (see Gorton and Metrick (2012a,b) and Copeland, Martin, and Walker (2011 (revised 2012)). Many financial institutions use collateralized borrowing as a source of financing. Some of them are private depository institutions, credit unions, mortgage real estate investment trusts, and security brokers and dealers.

Existing theories of collateralized debt rely on the borrower not being able to sell the collateral good or on the borrower exogenously valuing the asset more than lenders for collateralized debt to be optimal. However, and crucially, the assets that are offered as collateral in collateralized derivative trades, "sold" in repos or pledged by specialty lenders to raise funds are assets that could also be sold in financial markets. Moreover, in contrast to physical assets which are used as collateral to raise funds, such as family heirlooms that are pawned or real estate used to obtain credit lines, the intrinsic value of these financial assets is independent of the identity of the assets' holders. Then, why do so many financial institutions choose to use these financial assets as collateral instead of selling them to raise funds?

The main contribution of this paper is to provide a microfoundation for the use of financial assets as collateral by focusing on the borrower's optimal financing choice. I show that borrowing firms choose to collateralize financial assets (rather than selling them) when the return on their investment is not observable to the lenders, the asset is not perfectly liquid, and investment opportunities are persistent. The non-observability of the investment returns implies that asset transfers, either in the form of asset sales or collateralized debt, are necessary for the firm to raise funds and invest. Since financial assets allow firms to take advantage of their investment opportunities, borrower firms value the financial asset relatively more than the lenders. Therefore, when investment opportunities are persistent, borrowers want to start the next period with as many assets as possible. If the financial asset is perfectly liquid, the borrower is indifferent between asset sales and collateralized debt since he can always repurchase the asset he transfers to the lender in the asset market. When the financial asset is not perfectly liquid at the end of the period, borrowers maximize the expected asset holdings with which they start the next period by financing their investments with collateralized debt. When the asset is used as collateral, borrowers only lose the asset when their projects are not successful and there is default, rather than with certainty as it is the case with asset sales.

I develop these insights in a dynamic model of optimal funding. In the baseline version of my model there are two periods, each divided into two subperiods: morning and afternoon. There are two risk-neutral agents, a borrower (firm) and a lender, a storable consumption good and one long-term financial asset which pays dividends each afternoon. Each morning, the borrower can invest in a risky project that pays a random return in the afternoon of the same period. This project is profitable in expectation but it incurs losses if it is not successful. The borrower has his investment opportunity before the financial asset's dividends are paid and, thus, cannot invest without external financing.

The main friction in the model is that the return on the risky project is private information of the agent who invested in it. Because the return of the risky project is not observable and the project may incur losses, the borrower cannot issue unsecured debt and he needs the asset to raise funds to invest. Therefore, the borrower values the financial asset more than the lender because it allows him to take advantage of his investment opportunity. If investment opportunities (and thus the roles as borrower and lender) are persistent, as it is the case in many collateralized debt markets, the borrower tries to bring as much of the asset as possible to the next morning to raise funds and invest.[1] When the market for the asset is closed in the afternoon, as in the baseline model, a borrower cannot acquire new assets in the afternoon and, thus, he tries to retain the asset to the extent possible. The borrower can achieve this by financing his investment with collateralized debt, where the asset is only seized when the project's return is low and not transferred with probability one as it is the case in outright asset sales.

Intuitively, when the borrower values the asset relatively more than the lender in the afternoon, transferring the asset to the lender is costly for the borrower. For each unit of the asset the borrower transfers to the lender, the borrower gets at most the lender's valuation, which is less than his own. Alternatively, to raise the same amount of funds, the borrower can give the lender consumption good in an amount equal to the lender's valuation of the asset. Since the lender's valuation of the asset (in terms of the consumption good) is lower than the borrower's, the borrower finds it less expensive to increase his funding by transferring consumption goods to the lender rather than assets. Using the asset as collateral allows the borrower to transfer the asset only in the states in which he does not have enough resources to repay the lender in consumption good.

[1] This persistence is consistent with the observation that, in many collateralized debt markets, different types of institutions specialize in borrowing or lending. For example, in the repo market, money market funds are usually lenders, whereas hedge funds and specialty lenders are usually borrowers.

The difference in asset valuations between the borrower and the lender in the afternoon, which is crucial for collateralized debt to be optimal, arises endogenously and it depends on the asset being less than perfectly liquid in the afternoon. In Section 3, I extend the baseline model and consider the case in which the agents have probabilistic access to a competitive asset market in the afternoon of the first period. When the asset is perfectly liquid, and the agents can access the asset market in the afternoon with probability one, the marginal rates of substitution between the asset and the consumption good for borrowers and lenders are equalized and the borrower is indifferent between collateralized debt and asset sales. When the asset is less than perfectly liquid, and there is some probability that the borrower will not be able to access the asset market in the afternoon, the borrower values the asset relatively more than the lender and he strictly prefers collateralized debt over asset sales.

The value of the asset for borrowers is maximized when the asset is perfectly liquid and the asset market in the afternoon is frictionless. There is a loss of value, an *illiquidity discount*, associated with the asset's illiquidity in the afternoon. Borrowers are the efficient holders of the asset at the end of the first period because they can use the asset to raise funds to invest the next morning. When the asset is not perfectly liquid, some assets remain in the hands of lenders and the asset is inefficiently allocated at the end of the first period. This misallocation implies that borrowers start the next period with fewer assets and, thus, can invest less in their projects in the second period, which reduces the value of holding the asset to begin with. Since this misallocation increases with the asset's illiquidity, the illiquidity discount is higher when the asset is more illiquid.

The asset's illiquidity also affects the asset's value as collateral. This *collateral premium* is given by the borrower's valuation of the extra amount a borrower can get from the lender by pledging the asset as collateral instead of selling it. As I mentioned above, when the asset is perfectly liquid in the afternoon, the borrower is indifferent between using the asset as collateral and selling it outright. In this case, the collateral premium is zero. However, when the asset is illiquid, transferring the asset to the lender is costly for the borrower and the borrower can credibly commit to repaying the lender more than the lender's valuation in consumption goods to avoid losing the asset. This collateral premium depends on the difference in valuation between borrowers and lenders in the afternoon of the first period and it is increasing in the asset's illiquidity.

For borrowers to value the asset more in the afternoon, it is crucial that they expect to have a better use of the asset the next morning. In the first two models, borrowers value the asset more than lenders every morning since they always have an investment opportunity at the beginning of a period. To highlight this assumption, I extend the model with the asset market by making the access to investment opportunities stochastic. I show that as long as the access to investment opportunities is persistent, borrowers will have a higher valuation of the asset in the afternoon and a less than perfectly liquid asset will optimally be used as collateral.

Finally, I extend the model with stochastic investment opportunities and probabilistic access to the asset market in the afternoon to an infinite horizon model. Consistent with the results in the two-period models, when investment opportunities are persistent, a borrower strictly prefers to use the financial asset as collateral as long as the asset is not perfectly liquid and he is indifferent between asset sales and collateralized debt if the asset is perfectly liquid. The allocative efficiency of the stationary economy is increasing in the asset's liquidity because when the asset's liquidity in the afternoon is greater, the borrowers carry a larger fraction of the asset to the next period, and the total output in the economy is higher.

As I mentioned above, the main contribution of this paper microfound the use of financial assets as collateral by focusing on the borrower's optimal financing choice. The paper is related to the large literature concerning the optimal financing of investment projects. For example, in a setup with asymmetric information about the quality of a productive asset, Gorton and Ordoñez (2014) show that short-term collateralized debt is optimal if lenders are willing to accept the asset as collateral without acquiring information about it. They show that this is the case if the prior about the quality of the asset used as collateral is high enough. My paper differs from Gorton and Ordoñez (2014) in three ways: (1) I consider financial assets (as opposed to physical productive ones); (2) I allow for asset sales, which they do not consider; and (3) there is perfect information about the quality of the asset in my model.

Within the literature on optimal financing, this paper is closely related to papers that consider incomplete contracts, either due to private information or moral hazard, in which the optimal contract features collateralized debt. In Bolton and Scharfstein (1990), the optimal contract is a long-term contract in which the lender commits to liquidate the borrower's firm if default occurs. In Lacker (2001), non-pecuniary costs of default are assumed while in Rampini (2005), default penalties are modeled as transfers of goods only valued by the borrower. Gale and Hellwig (1985), Bernanke and Gertler (1989) and Bernanke, Gertler, and Gilchrist (1998) also find optimal contracts that resemble collateralized debt. They follow Townsend (1979) and allow for costly state verification to partially resolve the agency problems. In all of these papers, including my paper, the optimal contract involves some credible punishment for the borrower if the bad state is realized, which incentivizes him to behave, and report the state truthfully. The main contribution of my paper is that the cost of default for the borrower, which is the key determinant of the optimality of collateralized debt, arises endogenously and is not assumed.

In my model, contracts are incomplete but perfectly enforceable.[2] However, as Barro (1976) shows, enforcement frictions can also give rise to collateral contracts in equilibrium. Along these lines, Stiglitz and Weiss (1981), Chan and Kanatas (1985), and Kocherlakota (2001) analyze collateral as a mechanism to enforce contracts and deter default. Kiyotaki and Moore (1997) explore the effect of collateral constraints on the business cycle, while Rampini and Viswanathan (2010) and Rampini and Viswanathan (2013) examine the capital structure of the firm. In most of these papers, the assets that are used as collateral are either illiquid or they are inputs of production, so selling them to raise funds is not an option. In contrast, in my paper the asset that is used as collateral in equilibrium is a liquid financial asset.

One of the main advantages of deriving collateralized debt as part of the optimal contract is that the amount that can be borrowed against the asset, its debt capacity, is an equilibrium outcome. In this regard, my paper is also related to the endogenous leverage literature based on the general equilibrium models developed in Araujo, Orrillo, and Pascoa (1994), Geanakoplos (2003a,b), Geanakoplos and Zame (1997), and Fostel and Geanakoplos (2008). In these models, the collateral requirements are also equilibrium outcomes. However, in contrast to my paper, the contract space is restricted to collateralized debt in these papers.

The mechanism I describe in this paper can be applied to multiple types of debt collateralized by financial assets, including repurchase agreements. Other papers that look specifically at repo markets are Duffie (1996), Monnet and Narajabad (2012) and Gottardi, Maurin, and Monnet (2016). Duffie (1996) argues that differences in the transaction costs between selling an asset and entering a repurchase agreement justify the use of the latter. Monnet and Narajabad (2012) look at a search model in which agents prefer to conduct repurchase agreements than asset sales when they face substantial uncertainty about the exogenous private value of holding the asset in the future. In Gottardi, Maurin, and Monnet (2016), repos allow the borrowers to insure the initial buyer of the asset against price fluctuations.

The rest of the paper is organized as follows. In Section 2, I present the baseline two-period model and highlight the mechanism behind the main result. In Section 3, I expand the baseline

[2] Hart and Moore (1985) study the optimal contract in the presence of incomplete contracts that can be renegotiated.

model by considering the possibility of a competitive asset market. Section 4 extends the model in Section 3 to a dynamic economy. I conclude in Section 5. All omitted proofs are in the Appendix.

2 Baseline model

In this section, I present a baseline two-period model that highlights the forces behind the main result that borrowers strictly prefer to use financial assets as collateral rather than selling them to raise funds. This baseline model emphasizes the sources of the differences in valuation between borrowers and lenders that is at the core of the mechanism that makes collateralized debt optimal.

2.1 Environment

There are two periods, $t = 1, 2$, each divided into two subperiods: morning and afternoon. There are also two risk-neutral agents, a borrower (B) and a lender (L), with a common discount factor, $\beta \in (0, 1)$. There is one storable consumption good and one long-lived financial asset in the economy. a units of the financial asset yield da units of consumption good as a dividend in the afternoon of each period t. The lender is endowed with a large amount of consumption good $\bar{e}$ in each period t. The borrower is endowed with a_1^B units of the financial asset in the first period and has no consumption good endowment. Instead, in each period, the borrower has access to a risky constant return, short-term project. One unit of the consumption good invested in the project in the morning of period t yields a random payoff $\theta_t \in \{\theta_L, \theta_H\}$ in the afternoon of period t, where $\theta_L < \theta_H$. The risky project's realized payoff is only observed by the borrower who invested in it. The projects' returns are i.i.d. over time, where the probability distribution of the returns is given by $\pi_i \equiv \Pr(\theta_t = \theta_i)$ for $i = L, H$. Though the risky project is profitable in expectation, $1 < \mathbb{E}(\theta_t) \equiv \pi_L\theta_L + \pi_H\theta_H$, it incurs losses if the low state is realized, $0 \leq \theta_L < 1$. I denote by high (low) state the state in which $\theta_t = \theta_H$ ($\theta_t = \theta_L$). To simplify the analysis, I'll assume that $\theta_L = 0$.[3]

Assumption 1 (Impatience) The discount factor β satisfies $\beta\mathbb{E}(\theta) < 1$.

Assumption 1 implies that the borrower always chooses to consume the proceeds of the risky project rather than saving them to invest in his project in the future. The borrower can transfer wealth between periods either by storing consumption good or by holding the financial asset. Suppose that the borrower stores a unit of the consumption good to invest it in the risky project the

[3] All the results in the paper hold if $\theta_L \in (0, 1)$ with the addition that the implementation of the optimal contract would also include non-contingent, unsecured debt. In this case, Assumption 1 becomes $\beta\frac{\mathbb{E}(\theta)-\theta_L}{1-\theta_L} < 1$.

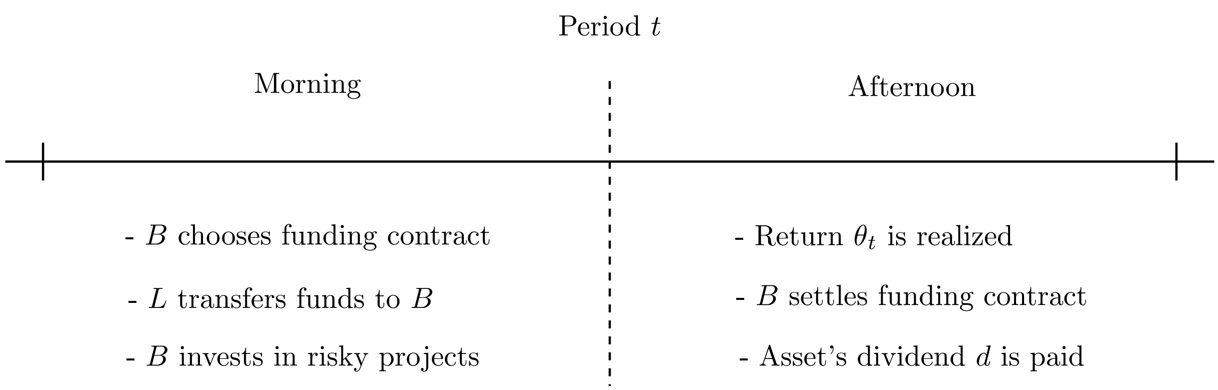

Figure 1: Timing

following morning. In this case, he expects to consume $\mathbb{E}(\theta)$ in the afternoon of the following period. Then, the borrower values saving one unit of consumption in $\beta\mathbb{E}(\theta)$. However, given Assumption 1, the borrower finds it suboptimal to store consumption good and will always choose to begin period 2 with no consumption good. Since borrowers have a better use for savings ($\mathbb{E}(\theta) > 1$), if the borrower chooses not save, the lender will choose not to save either. Therefore, I can ignore the agents' consumption good saving decisions in the remainder of the paper. Finally, since the agents' would rather consume than save, Assumption 1 implies that the asset forces its holder to save.

The morning of each period t, the borrower and the lender enter into a short-term funding contract in which all bargaining power is given to the borrower. Immediately after the terms of the funding contract have been determined, the borrower invests in his risky project. In the afternoon of date t, the return of the time t investment is realized, the terms of the contract are settled, and the financial asset's dividend is paid. This timing is depicted in Figure 1.

The (financial) asset's dividends are paid in the afternoon, while the investment is made in the morning. This timing has two important implications. First, if left in autarky, the borrower and the lender value the asset equally as the discounted sum of its dividends. There is no intrinsic difference in asset valuations between the borrower and the lender. Second, the borrower needs to get funds from the lender to take advantage of his investment opportunity.

Definition 1 (Short-term funding contract) *A short-term funding contract consists of a funding amount in terms of consumption good, k, contingent repayments in terms of consumption good, r_i, and in terms of asset transfers, t_i, $i = L, H$. Repayments in terms of assets include the transfer* *of the contemporaneous asset's dividends.*[4]

In the context of the funding contract in Definition 1, I will interpret non-contingent asset transfers as asset sales and contingent asset transfers as collateralized loans in which the asset is transferred if default occurs. More specifically, when $t_L \neq t_H$, part of the asset transfers are contingent and the funding contract can be implemented using collateralized loans, as follows. If $t_L < (>) \, t_H$, the repayment of the collateralized part of the loan is $|r_H - r_L|$ in terms of consumption goods in the high (low) state, when there is no default, and $|t_H - t_L|$ in terms of asset transfers in the low (high) state, when default occurs.

The set of transfers implied by any contract to which the borrower and lender agree has to be feasible given the resources held by both parties.

Definition 2 (Feasibility) *A short-term contract* (k, r_L, r_H, t_L, t_H) *is feasible in period* t *if*

$$
\begin{aligned}
&0 \leq k \leq \bar{e} \\
&0 \leq r_i \leq d\,(a_t - t_i) + \theta_i k \quad && i = L, H \\
&0 \leq t_i \leq a_t && i = L, H
\end{aligned}
$$

where a_t *is the amount of the asset held by the borrower in period* t*.*

The first constraint in Definition 2 states that the funding amount k has to be non-negative and that it cannot be larger than the lender's endowment of the consumption good. The second set of constraints implies that the state contingent repayments in terms of consumption good cannot be negative, nor can they be more than the amount of the good the borrower has at the end of the period in each state after the asset transfer t_i. Similarly, the third set of constraints implies that the borrower cannot transfer more assets than the amount he holds and does not allow the borrower to purchase assets from the lender at the time of settlement. In the model developed in Section 3, I consider the case in which the borrower can use the proceeds of this investment opportunity to purchase assets.

The non-observability of the returns of the risky project is the key friction that gives rise to collateralized debt in equilibrium. Since the return of the risky project is only observed by the borrower who invested in it and is not verifiable by the lender, the loan contract between the lender and the borrower cannot be contingent on the realization of the return, θ_t. However, a contract can be made contingent on the reported return of the risky project as long as it is incentive compatible.

[4] This assumption is without loss of generality since the dividends are observable, and thus pledgeable.

Definition 3 (Incentive Compatibility) *A contract* (k, r_L, r_H, t_L, t_H) *is incentive compatible in period* t *if whenever* $r_L \leq \theta_H k + d(a - t_L)$

$$-r_L - dt_L + \beta V_{t+1}^B(a - t_L) \leq -r_H - dt_H + \beta V_{t+1}^B(a - t_H) \qquad (IC_H)$$

and whenever $r_H \leq \theta_L k + d(a - t_H)$

$$-r_H - dt_H + \beta V_{t+1}^B(a - t_H) \leq -r_L - dt_L + \beta V_{t+1}^B(a - t_L) \qquad (IC_L)$$

where $V_{t+1}^B(a)$ *is the value of* a *units of the financial asset to the borrower in period* $t+1$*.*

If the borrower reports state i as being realized, he transfers r_i units of the consumption good and t_i units of the financial asset to the lender. Since the transfer of the asset also implies the transfer of the asset's dividends, transferring t_i units of the asset decreases the borrower's current consumption by dt_i, on top of reducing the stock of capital with which the borrower begins the following period.

The first constraint IC_H states that, in the high state, it is always at least as good for the borrower to tell the truth and report that the high state has occurred than to lie and report a low realization of θ. The second constraint IC_L states the analogous for the low state. Note that the constraints IC_H and IC_L are only active when lying is feasible (i.e., when there are enough resources in the low (high) state to match the contingent repayment in terms of goods promised in the high (low) state, r_H (r_L)).

Three features of the model are worth emphasizing:

Financial assets. The long-lived asset a is a financial asset. It is not related to the project available to the borrower: the asset is not an input needed to operate the project. This assumption differentiates this paper from other papers of collateral where the asset used as collateral is a physical asset; in such cases, the asset is physically needed to operate the borrower's technology and, thus, cannot be sold without forgoing access to the investment opportunity.

Asset sales and liquidity. Though there is no formal market for the asset in the mornings, the borrower can always sell his asset holdings to the lender at a price equal to the present value of its dividends by setting $t_L = t_H = a$ and $r_L = r_H = 0$, which satisfies feasibility and incentive compatibility. Therefore, in the morning, the durable financial asset is liquid—it can be sold at the time at which the borrower makes his financing decision. However, in the baseline model, the asset is *perfectly illiquid* in the afternoon. As I show in Section 3, it is crucial for the financial asset to be less than perfectly liquid for collateralized debt to be strictly preferred over asset sales.

Contracts. The funding contracts between the borrower and the lender are short-term, complete contracts subject to incentive compatibility constraints. The main friction in the model is that cash flows are not observable. Therefore, contracts can only depend on the borrower's reports of the return of the projects. Though I restrict the analysis in the paper to short-term contracts, I show in the Appendix that short-term contracts can implement the optimal long-term contract subject to incentive compatibility constraints, when the lender has limited commitment.

2.2 Equilibrium

In this section, I compute the equilibrium of the baseline model. I start by looking at the last period and show that the borrower values the asset more than the lender in the morning of the second period because it allows him to raise funds to invest in his risky project. Then, I look at the equilibrium in the first period and get to the main result that borrowers strictly prefer to use financial assets as collateral rather than selling them to raise funds.

2.2.1 Second period

In the morning of the second period, $t = 2$, the borrower chooses a feasible, incentive compatible, short-term contract to maximize his expected consumption. He invests the loan amount k in his risky project, which yields an expected return $\mathbb{E}(\theta)\,k$. The borrower expects to repay $\pi_L r_L + \pi_H r_H$ in terms of the consumption good and to transfer t_i units of the financial asset if state $i = L, H$ is realized. Finally, the borrower also gets the dividends paid by his asset holdings, da. Thus, a borrower with assets a solves the following problem in period 2

$$V_2^B(a) = \max_{k, r_L, r_H, t_H, t_L} \mathbb{E}(\theta)\,k - \pi_L(r_L + dt_L) - \pi_H(r_H + dt_H) + da,$$

subject to the short-term funding contract (k, r_L, r_H, t_H, t_L) being feasible and incentive compatible, and satisfying the lender's participation constraint

$$k \leq \pi_L(r_L + dt_L) + \pi_H(r_H + dt_H).$$

Proposition 1 *The equilibrium short-term funding contract in period 2 is given by*

$$\begin{aligned} k_2^*(a) &= da, \\ r_{2L}^*(a) &= 0, \quad t_{2L}^*(a) = a, \ \textit{and} \\ r_{2H}^*(a) + dt_{2H}^*(a) &= k_2^*(a) \quad \textit{for any} \ \ t_{2H}^*(a) \in [0, a]. \end{aligned}$$

The proof of Proposition 1 is straightforward. Since the asset has no value after the second period, incentive compatibility in the second period implies that the funding contract is risk-free for the lender, i.e., $r_L + dt_L = r_H + dt_H$. Moreover, the borrower promises to repay the maximum amount he can credibly commit to repaying in the low state, i.e., $r_L + dt_L = da$. This, together with the participation constraint for the lender, implies

$$k_2^*(a) = \pi_L(r_{2L}^*(a) + dt_{2L}^*(a)) + \pi_H(r_{2H}^*(a) + dt_{2H}^*(a)) = da.$$

As Proposition 1 states, there are many transfer schemes through which the borrower can implement the optimal funding contract. In particular, the borrower is indifferent between collateralized debt and asset sales.

Corollary 1 *In the second period, the borrower is indifferent between selling his assets and pledging them as collateral.*

Since $\theta_L = 0$, the borrower cannot commit to unsecured debt at all. However, the loan amount $k_2^*(a)$ can be thought of either as the proceeds from asset sales, i.e., setting $t_{2L}^* = t_{2H}^* = a$, or as debt collateralized by a units of the asset, i.e., setting $t_{2L}^* = a$ and $t_{2H}^* = 0$. This shows that the borrower is indifferent between selling his assets and pledging them as collateral in the second period.

Differences in valuation For a lender, the value of holding a units of the asset in the morning of the second period is

$$V_2^L(a) = da.$$

Because a lender does not have access to investment opportunities, he can only consume the dividends paid by his asset holdings. Therefore, the lender values the asset for the dividends it pays.

For a borrower, the value of holding assets a in the morning of the second period is

$$V_2^B(a) = \mathbb{E}(\theta)\, da > da = V_2^L(a).$$

In the morning of the second period, before investment takes place, the borrower values the asset more than the dividends it pays because having the asset allows him to raise funds and take advantage of his investment opportunity. Since the return of the risky project is not observable, without the asset, the borrower would not be able to commit to repaying the lender (recall $\theta_L = 0$), and, therefore, the lender would never agree to providing an unsecured loan. Having the asset allows the borrower to commit to transferring the asset's dividends, either by selling the asset or by using it as collateral. In this case, even if the borrower always reports the low state being realized, the lender can collect da. Thus, holding a units of the asset allows the borrower to invest

$$k_2^* (a) = da$$

in the morning of the second period, on which he expects to earn a return of $\mathbb{E}(\theta)$.

The borrower's marginal valuation of the asset in the morning of period 2 is $v_2^B = \mathbb{E}(\theta)\, d$, which is greater than the lender's marginal valuation of the asset, $v_2^L = d$. The extra value the borrower attaches to the asset is the additional value the borrower gets from being able to raise funds against the asset in the morning and investing the proceeds from the sale in his risky project. This difference is given by the net return of the borrower's risky project,

$$v_2^B - d = (\mathbb{E}(\theta) - 1)\, d.$$

2.2.2 First period

Analogous to the problem in the morning of the second period, in the morning of the first period the borrower solves

$$V_1^B (a) = \max_{k, r_L, r_H, t_H, t_L} \mathbb{E}(\theta)\, k - \pi_L (r_L + dt_L) - \pi_H (r_H + dt_H) + da + \pi_L \beta V_2^B (a - t_L) + \pi_H \beta V_2^B (a - t_H),$$

subject to the short-term funding contract (k, r_L, r_H, t_H, t_L) being feasible and incentive compatible and satisfying the lender's participation constraint:

$$k \leq \pi_L \left(r_L + dt_L + \beta V_2^L (t_L) \right) + \pi_H \left(r_H + dt_H + \beta V_2^L (t_H) \right).$$

Proposition 2 *The equilibrium short-term funding contract in period* 1 *is given by*

$$\begin{aligned} &k_1^* (a) = (1 + \beta)\, da + \beta \pi_H \left(V_2^B (a) - V_2^L (a) \right) \\ &t_{1L}^* (a) = a, \qquad r_{1L}^* (a) = 0, \\ &t_{1H}^* (a) = 0, \ \textit{and} \quad r_{1H}^* (a) = da + \beta V_2^B (a). \end{aligned}$$

As the proposition above states, the total size of the loan the borrower can procure from the lender in the first period depends on the resources he can credibly commit to transferring to the lender. By choosing to sell his assets and setting $t_{1L}(a) = t_{1H}(a) = a$, the borrower can indirectly pledge the assets' dividends in periods 1 and 2. Selling the asset allows the borrower to increase the loan size by $d + \beta v_2^L$ per unit of asset sold, where $v_2^L = d$ is the lender's marginal valuation for the asset in the morning of period 2.

However, transferring the asset to the lender is costly for the borrower in the afternoon of the first period. For each unit of the asset the borrower transfers to the lender, the lender is willing to lend the borrower $d + \beta v_2^L$, which is the lender's valuation for the asset (cum dividend). The cost of this transaction for the borrower is $d + \beta v_2^B$. Alternatively, the borrower can increase the size of the loan in the same amount by transferring the lender $d + \beta v_2^L$ units of consumption good in the afternoon, which the borrower values $d + \beta v_2^L$. Since $v_2^B = \mathbb{E}(\theta) d > d = v_2^L$, the borrower would rather repay the lender in consumption good than in assets whenever possible and transfer as few units of the asset to the lender as possible. Setting $t_{1L} = a$, $t_{1H} = 0$ and $r_{1H} > r_{1L} = 0$ allows him to achieve that. The difference in marginal valuations referred to above allows the borrower to credibly commit to a state-contingent contract by effectively pledging part of the non-observable return of his project if the high state is realized.

Whenever the borrower values the asset relatively higher than the lender at the time of the contract's settlement, the borrower finds it less expensive to repay the lender in consumption good than in assets. In other words, whenever the marginal rate of substitution between the asset and the consumption good is higher for the borrower than for the lender, transferring the asset is costly for the borrower, and the borrower strictly prefers pledging the asset as collateral rather than selling it.

The additional amount the borrower is willing to pay in terms of consumption good in order not to loose the asset in the high state is $\beta \left(V_2^B(a) - V_2^L(a)\right)$. Then, the extra amount the borrower can get from the lender by pledging the asset as collateral instead of selling it is

$$\Delta k^* = \beta \pi_H \left(V_2^B(a) - V_2^L(a)\right) > 0.$$

Corollary 2 *In period* 1*, the borrower strictly prefers to pledge his assets as collateral rather than selling them.*

Equivalent to the implementation of the optimal contract at $t = 2$, the borrower cannot issue any unsecured debt since $\theta_L = 0$. The loan amount, $k_1^*(a)$, is repaid in consumption good only if the return of the project is high, whereas if the return is low, the lender receives a units of the asset from the borrower. Therefore, $k_1^*(a)$ can be interpreted as a loan collateralized by a units of the asset at an interest rate of:

$$i^c = \frac{r^*_{1H}(a)}{k^*_1(a)} - 1 = \frac{\beta\pi_L\left(V^B_2(a) - V^L_2(a)\right)}{da + \beta V^L_2(a) + \beta\pi_H\left(V^B_2(a) - V^L_2(a)\right)}.$$

The higher the difference in asset valuations between the borrower and the lender, the higher the amount of consumption goods the borrower can credibly commit to repay in the high state by pledging his assets as collateral.

The maximum amount that the borrower can obtain against one unit of the asset, its debt capacity, is given by:

$$D = \underbrace{(1+\beta)\,d}_{\text{Insurance}} + \underbrace{\beta\pi_H\left(v^B_2 - v^L_2\right)}_{\text{Incentives}},$$

where, as defined above, v^B_2 and v^L_2 are the borrower's and lender's marginal valuations of the asset, respectively.[5]

The asset's debt capacity depends on its value as collateral for lenders when there is default (insurance) and on the extra value borrowers attach to the asset when there is no default (incentives). Ceteris paribus, a higher asset value for lenders increases the loan amount since it allows lenders to recover more in the event of default, while a higher (extra) asset value for borrowers decreases the borrower's incentives to report untruthfully and, therefore, allows them to borrow more against the asset.

Collateralized debt and savings. Given Assumption 1, neither the borrower nor the lender likes to save. If they could consume their asset holdings, they would—$1 > \beta\mathbb{E}(\theta)$. However, since the asset is not a consumption good, the asset forces its holder to save. Moreover, savings are more valuable in the hands of borrowers than in the hands of lenders, because borrowers have access to profitable investment opportunities. Collateralized debt maximizes the amount of savings in the hands of borrowers by transferring as little of the asset to the lender as possible and, therefore, is optimal.

Asset's illiquidity. The difference in marginal rates of substitution that leads borrowers to strictly prefer to use collateral debt over asset sales relies on the asset not being perfectly liquid in the afternoon of the first period. If the asset is perfectly liquid and the borrower can use the proceeds from a successful project to purchase assets, the borrower finds it equally costly to transfer consumption good or the asset to the lender. In this case, the borrower is indifferent between using

[5] Since preferences are quasilinear in consumption and there is no asset market in the afternoon, the marginal valuations for the asset are also the marginal rate of substitution between the asset and consumption good.

the asset as collateral and selling it to raise funds. The model in Section 3 derives these results by considering the case in which an asset market may be available at the end of the first period.

Valuations for the asset The lender's value of holding a units of the asset in the morning of the first period is

$$V_1^L(a) = (1+\beta)\, da.$$

Since the lender cannot extract any surplus from the borrower, he can only consume the asset's dividends. Thus, the lender values the asset as its discounted sum of dividends.

The borrower's value of holding a units of the asset in the morning of the first period, $V_1^B(a)$, is

$$V_1^B(a) = \mathbb{E}(\theta)\, k_1^*(a),$$

where

$$k_1^*(a) = (1+\beta+\beta\pi_H(\mathbb{E}(\theta)-1))\, da.$$

Under the optimal funding contract, the borrower only values the asset as if he lived only one period. In the optimal contract, the borrower only gets to keep the asset if the high state is realized. However, incentive compatibility requires the borrower to transfer consumption goods in the same amount as the borrower's valuation for the asset if the high state is reported. Therefore, the borrower cares about the expected return $\mathbb{E}(\theta)$ of the current investment $k_1^*(a)$.

The borrower gets an expected return of $\mathbb{E}(\theta)$ per unit invested in his project. Holding a units of the asset allows him to invest $k_1^*(a)$ under the optimal funding contract. By setting $t_{1L} = t_{1H} = a$, the borrower can sell the asset to the lender at $t = 1$ in exchange for the lender's valuation, $da+\beta da$. Therefore, asset sales allow the borrower to invest $da + \beta da$ in his project. By pledging the asset as collateral, the borrower can invest an additional amount, $\Delta k^* = \beta\pi_H(\mathbb{E}(\theta)-1)\, da$.

3 Market for the asset

In the baseline model, it is crucial for the borrower and the lender to value the asset differently for the borrower to strictly prefer to use the asset as collateral. This difference in marginal rates of substitutions between the asset and the consumption good, which is at the center of the main result, depends on how easy it is for the borrower and the lender to transform consumption goods into assets and vice-versa at the end of the first period. The model presented in this section extends the baseline model to consider the case in which, with certain probability, the borrower and the lender can access a competitive asset market in the afternoon of the first period. The asset's market liquidity, that is, the probability with which an agent can access the asset market, is a crucial determinant of the valuations for the asset.

The model is the same as the baseline two-period model described in Section 2 with two differences. First, there is a continuum of borrowers and lenders, ex-ante identical within type. Second, and most importantly, with probability α, each agent can trade in a competitive asset market in the afternoon of $t = 1$, after the funding contracts have been settled.

As in the baseline model, in the morning of $t = 2$ borrowers are indifferent between selling their assets or pledging them as collateral. Thus, a borrower's valuation for a units of the asset in the morning of $t = 2$ is

$$V_2^B(a) = \mathbb{E}(\theta)\, da,$$

while a lender values a units of the asset $V_2^L(a) = da$.

3.1 Asset market

Suppose a borrower can trade in the asset market in the afternoon of $t = 1$, which occurs with probability α. Then, a borrower with assets a and wealth w will buy x units of the asset to solve the following problem:

$$V_{mkt}^B(a, w) = \max_x \beta V_2^B(a + x) + w - qx, \tag{1}$$

subject to

$$\begin{aligned} qx &\leq w \\ x &\geq -a, \end{aligned}$$

where q is the price of the asset in terms of consumption good. The first constraint above is the borrower's budget constraint: he cannot pay more for his purchases than the resources he has. The second constraint above implies that the borrower cannot sell more units of the asset than the ones he owns. The wealth level w is the residual project return after paying the lender.

A borrower with assets a and wealth w has a demand for the asset given by

$$x^B(a, w) = \begin{cases} -a & \text{if } q > \beta v_2^B \\ x \in \left[-a, \frac{w}{q}\right] & \text{if } q = \beta v_2^B \\ \frac{w}{q} & \text{if } q < \beta v_2^B \end{cases}.$$

Given the linear structure of the problem in (1), a borrower will (i) sell all of his assets if the asset price q is above his marginal valuation for the asset; (ii) he will buy as many assets as he can if his marginal valuation for the asset is greater than the price; and (iii) he will be indifferent between buying or selling otherwise.

If the aggregate assets held by borrowers are $\bar{a}_B$ and their aggregate wealth is $\bar{w}_B$, the borrowers' aggregate demand in the asset market is

$$X^B\left(\bar{a}^B, \bar{w}^B\right) = \begin{cases} -\alpha\bar{a}^B & \text{if } q > \beta v_2^B \\ x \in \left[-\alpha\bar{a}^B, \frac{\alpha\bar{w}^B}{q}\right] & \text{if } q = \beta v_2^B \\ \frac{\alpha\bar{w}^B}{q} & \text{if } q < \beta v_2^B \end{cases} . \tag{2}$$

Similarly, a lender with assets a and wealth w has a demand for the asset given by:

$$x^L = \begin{cases} -a & \text{if } q > \beta v_2^L \\ x \in \left[-a, \frac{w}{q}\right] & \text{if } q = \beta v_2^L \\ \frac{w}{q} & \text{if } q < \beta v_2^L \end{cases}$$

and the lenders' aggregate demand in the asset market is:

$$X^L\left(\bar{a}^L, \bar{w}^L\right) = \begin{cases} -\alpha\bar{a}^L & \text{if } q > \beta v_2^L \\ x \in \left[-\alpha\bar{a}^L, \frac{\alpha\bar{w}^L}{q}\right] & \text{if } q = \beta v_2^L \\ \frac{\alpha\bar{w}^L}{q} & \text{if } q < \beta v_2^L \end{cases} , \tag{3}$$

where $\bar{a}^L$ is the lenders' aggregate asset holdings and $\bar{w}^L$ is their aggregate wealth.

Since $v_2^B > v_2^L$, borrowers are buyers in the asset market and lenders are sellers. Moreover, because there is a fixed supply of assets and a fixed supply of goods, the price in the competitive asset market is determined by cash-in-the-market pricing. Thus, the equilibrium asset price q^* is given by:

$$q^* = \max\left\{\beta v_2^L, \min\left\{\frac{\bar{w}^B\left(\bar{a}^B\right)}{\bar{a}^L}, \beta v_2^B\right\}\right\}, \tag{4}$$

where

$$\bar{w}^B\left(\bar{a}^B\right) = \mathbb{E}\left(\theta\right)k^*\left(\bar{a}^B\right) - \pi_H r_H^*\left(\bar{a}^B\right) - \pi_L r_L^*\left(\bar{a}^B\right) + d\left(\bar{a}^B - \pi_H t_H^*\left(\bar{a}^B\right) - \pi_L t_L^*\left(\bar{a}^B\right)\right)$$

and $\left(k^*\left(\cdot\right), r_L^*\left(\cdot\right), r_H^*\left(\cdot\right), t_L^*\left(\cdot\right), t_H^*\left(\cdot\right)\right)$ is the optimal contract chosen by the borrowers in the funding market.

The equilibrium price q^* in Equation 4 depends on the liquidity available in the asset market, i.e., on the ratio between the available consumption goods held by the borrowers (buyers) and the available assets held by the lenders (sellers). The lower this ratio, the lower the equilibrium price q^*. For example, if $\frac{\bar{w}^B}{\bar{a}^L} < \min\left\{\beta d, \beta v_2^B\right\}$, there are more assets in the hands of lenders than what the borrowers are able to afford and the price in the asset market in the afternoon of the first period is $q^* = \beta d$. Alternatively, if $\bar{w}^B > \beta v_2^B \bar{a}^L$, lenders' have fewer assets than the amount the borrowers are able and willing to buy. In this case, the asset supply in the asset market is limited and the price is $q^* = \beta \mathbb{E}(\theta) d$.

Finally, if the asset market is open, the value of holding a units of the asset and wealth w for a borrower is

$$V_{mkt}^B(a, w) = \beta V_2^B \left(a + \frac{w}{q^*}\right)$$

while the value for a lender is

$$V_{mkt}^L(a, w) = q^* a + w.$$

Remark. *The asset market structure described above can also be interpreted as the result of a market in which there are search or matching frictions, as follows. Let α be the probability with which a borrower and a lender meet. Upon meeting, the terms of trade between them are determined through Nash bargaining, where the bargaining power of the borrower is $\gamma \in [0, 1]$. This market structure and the competitive one described above are isomorphic if the bargaining power of the lender and the borrower is chosen appropriately. More specifically, they are isomorphic if the bargaining power of the borrower is set to:*

$$\gamma = \frac{\beta v_2^B - q^*}{\beta\left(v_2^B - v_2^L\right)}.$$

3.2 Funding market

In the morning of the first period, the borrower chooses his funding contract taking into account that with probability α he will be able to use the proceeds from his project to purchase assets in the asset market.[6] Then, the borrower's expected value of holding a units of the asset and w units of the consumption good at the time of settlement in terms of $t = 1$ consumption is

$$\beta \hat{V}_2^B(a, w) \equiv (1 - \alpha)\left(w + \beta V_2^B(a)\right) + \alpha V_{mkt}^B(a, w). \tag{5}$$

When the funding contract is settled in the afternoon of the first period, the borrower has access to the asset market with probability α. In this case, the borrower values having a units of the asset and wealth w in $V_{mkt}^B(a, w)$. With probability $(1 - \alpha)$, the borrower cannot access the asset

[6] For simplicity, the funding contract cannot be contingent on whether the asset market is open.

market in the afternoon and he consumes his wealth and brings his assets a to the morning of the following period, which he values $V_2^B(a)$. This is summarized in Equation (5).

Similarly, the lender's expected value of holding a units of the asset at the time of settlement of $t = 1$ in terms of $t = 1$ consumption is

$$\beta \hat{V}_2^L(a) \equiv ((1-\alpha)\beta d + \alpha q^*) a. \tag{6}$$

The lender gets to sell his assets a at a price q^* with probability α and, with probability $(1-\alpha)$, he has to wait until the asset's dividends are paid at the end of the following period to consume. Equation (6) reflects this.

Let $w_i \equiv \theta_i k_1 - r_{1i} + d(a - t_{1i})$ be the borrower's wealth after settling his contract in the first period if the return of his project is θ_i. Then, in the morning of the first period, a borrower with assets a solves the following problem

$$V_1^B(a) = \max_{k_1, r_{1L}, r_{1H}. t_{1L}, t_{1H}} \pi_H \beta \hat{V}_2^B(a - t_H, w_H) + \pi_L \beta \hat{V}_2^B(a - t_L, w_L),$$

subject to the feasibility constraints in equation (2), the incentive compatibility constraint in the high state if lying is feasible, i.e., if $r_{1L} \leq \theta_H k_1 + d(a - t_{1H})$

$$\hat{V}_2^B(a - t_H, w_H) \geq \hat{V}_2^B(a - t_L, w_L + (\theta_H - \theta_L) k_1),$$

the incentive compatibility constraint in the low state if lying is feasible, i.e., if $r_{1H} \leq \theta_L k_1 + d(a - t_{1L})$,

$$\hat{V}_2^B(a - t_L, w_L) \geq \hat{V}_2^B(a - t_H, w_H + (\theta_L - \theta_H) k_1),$$

and the lender's participation constraint

$$k_1 \leq \pi_H(r_{1H} + dt_{1H}) + \pi_L(r_{1L} + dt_{1L}) + \beta\left(\pi_H \hat{V}_2^L(t_{1H}) + \pi_L \hat{V}_2^L(t_{1L})\right).$$

In the baseline model presented in Section 2, though the asset was liquid (could be sold) in the morning, it was perfectly illiquid in the afternoon. In the model presented in this section, the asset's market liquidity depends on the parameter α.

Definition 4 *An asset's market liquidity in the afternoon is determined by the likelihood with which it can be traded in the afternoon of $t = 1$, i.e., by the parameter α.*

As I mentioned in the analysis in Section 2, the parameter α, which from now on I will refer to as the asset's market liquidity, is a key determinant of the valuations for borrowers and lenders, and, thus, of the optimal funding contract. The following proposition formalizes this link.

Proposition 3 *The borrowers' optimal funding contract depends on the asset market's liquidity.*

a) Borrowers are indifferent between pledging the asset as collateral and selling it in the morning of the first period if and only if the asset is perfectly liquid in the afternoon, i.e., $\alpha = 1$*.*

b) Borrowers strictly prefer to pledge the asset as collateral rather than sell it in the morning of the first period if and only if the asset is not perfectly liquid in the afternoon, i.e., $\alpha \in [0, 1)$*.*

When the asset market is not available at the end of $t = 1$, i.e., $\alpha = 0$, the borrower prefers to repay the lender in terms of consumption good rather than in assets. In this case, as in the baseline model, the marginal rate of substitution between assets and consumption goods differs for borrowers and lenders. For each unit of the asset the borrower transfers to the lender, the lender is willing to increase the loan size by $d+\beta v_2^L$. This transfer costs the borrower $d+\beta v_2^B$. Alternatively, the borrower could get the lender to increase the size of the loan by $d+\beta v_2^L$ by transferring $d+\beta v_2^L$ units of consumption good, which would cost him only $d+\beta v_2^L$. Since $v_2^L < v_2^B$, it is less costly for the borrower to repay the lender in consumption goods than to repay him in terms of assets.

If the asset market is always available in the afternoon of $t = 1$, i.e., $\alpha = 1$, the borrower is indifferent between repaying the lender in terms of assets and consumption goods. For each unit of the asset he gives the lender, the lender is willing to increase the size of the loan in $d + q^*$. Alternatively, the borrower could attain the same increase in loan size by transferring $d + q^*$ units of consumption good to the lender. However, the borrower's opportunity cost of this transfer is not being able to buy one unit of the asset in the market. Therefore, transferring the lender one unit of the asset or $d + q^*$ units of consumption goods has the same cost for the borrower as they decrease the borrower's utility by $d + \beta v_2^B$.

When the asset market is randomly available, i.e., $\alpha \in (0, 1)$, in expectation it is less costly for the borrower to repay the lender in consumption goods than in assets. In other words, transferring the asset to the lender is costly for the borrower in expectation. Analogous to the baseline model, whenever transferring the asset to the lender is more costly for the borrower than transferring consumption goods, the borrower strictly prefers to pledge the asset as collateral and only transfer it to the lender if he cannot remit compensation in consumption goods.

Remark. *The liquidity of the asset at the end of the first period depends on* α*. However, as mentioned above, the asset is liquid at the beginning of the period since the borrower can always choose to sell his assets to the lender by setting* $t_L = t_H = a$*. As illustrated in this section, the borrower will choose to pledge his liquid financial asset to raise funds as long as the asset is not perfectly liquid at the end of the period.*

Assuming $\alpha < 1$ implies some asset buyers and sellers are excluded from the asset market, despite them being willing to trade. This is a reasonable assumption given the evidence of market freezes. Market freezes were recurrent during the last financial crisis. However, they can also occur in less extreme situations even in markets for the most liquid assets. For example, in July 2015, U.S. Treasury debt, which is widely used as collateral in repo markets, became increasingly harder to trade in response to binding capital constraints faced by banks. According to the Financial Times, "banks pulled back from quoting prices, while other traders reduced their presence during the most extreme moments of turmoil, impairing the ability of investors to transact cash bonds." Similarly, the liquidity in the U.S. Treasury debt market decreased notably in 2013 in anticipation of the tapering of the Federal Reserve's quantitative easing policy.[7]

More generally, any transaction cost or friction that imposes a wedge in the marginal rates of substitution between assets and consumption goods for the borrower and the lender will lead to the borrower strictly preferring to use the asset as collateral rather than sell it.

3.3 Liquidity

The ease with which an asset can be traded, that is, its market liquidity, is a key determinant of the equilibrium valuations for borrowers and lenders. In the model described in this section, the asset is liquid in the mornings, whereas the liquidity of the asset in the afternoon depends on the parameter α. If $\alpha = 1$, the asset is perfectly liquid, and if $\alpha < 1$, the asset is less than perfectly liquid. In this section, I show that more liquid assets are valued more by lenders and borrowers, and discuss the illiquidity discount and collateral premium associated with the financial asset.

3.3.1 Valuations for the asset and prices

In the morning of each period, the borrower can always sell the asset to the lender by setting $t_L = t_H$ at a price equal to the lenders' valuation for the asset, given by

$$v_1^L \equiv d + (1 - \alpha)\,\beta d + \alpha q^*. \tag{7}$$

If a lender holds one unit of the asset at the beginning of $t = 1$, he gets to consume the dividends of the asset in the afternoon of that period. With probability α the lender has access to the asset

[7] See "U.S. Treasuries market faces liquidity concerns", *Financial Times*, July 29, 2015, and "Liquidity deteriorates for U.S. Treasuries", *Financial Times*, November 23, 2015. Retrieved from http://www.ft.com/. Last accessed April 1st, 2016.

market and sells the asset for q^* units of the consumption good; and with probability $(1-\alpha)$, he doesn't have access to this market and consumes the dividends the asset pays in the afternoon of the following period.

The expected marginal utility of consumption for a borrower in the morning of period 1 is $(1-\alpha)+\alpha\frac{\beta\mathbb{E}(\theta)d}{q^*}$. With probability α, the borrower has access to the asset market in the afternoon and can buy assets, which he values $\beta\mathbb{E}(\theta)\,d$, for q^* units of the consumption good. With probability $(1-\alpha)$, the borrower cannot purchase new assets and chooses to consume his consumption good. Then, the borrowers' value (per asset) of selling the asset in the funding market in the morning of the first period is

$$v_{1s}^B \equiv \left((1-\alpha)+\alpha\frac{\beta\mathbb{E}(\theta)\,d}{q^*}\right)\mathbb{E}(\theta)\,v_1^L.$$

Under the optimal funding contract, the borrower can raise $\frac{k_1^*}{a}$ from the lender per unit of the asset he holds. Thus, a borrower's marginal valuation for the asset under the optimal contract is

$$v_1^B = \left((1-\alpha)+\alpha\frac{\beta\mathbb{E}(\theta)\,d}{q^*}\right)\mathbb{E}(\theta)\,\frac{k_1^*}{a}, \tag{8}$$

where k_1^* is the size of the loan in the optimal contract in the first period and is given by

$$k_1^* = da+\pi_L\left(\alpha q^*+(1-\alpha)\,\beta d\right)a+\pi_H\frac{\beta\mathbb{E}(\theta)\,d}{\left((1-\alpha)+\alpha\frac{\beta\mathbb{E}(\theta)d}{q^*}\right)}a.$$

In the afternoon, the borrower can buy the asset from lenders with probability α at a price q^* given by

$$q^* = \max\left\{\beta d, \min\left\{\frac{\bar{w}^B\left(\bar{a}^B\right)}{\bar{a}^L}, \beta\mathbb{E}(\theta)\right\}\right\} \tag{9}$$

where

$$\bar{w}^B\left(\bar{a}^B\right) = \left(\mathbb{E}(\theta)\left(d+\pi_L\left(\alpha\left(q^*-\beta d\right)+\beta d\right)\right)+\left(\mathbb{E}(\theta)-1\right)\pi_H\frac{\beta\mathbb{E}(\theta)}{(1-\alpha)+\alpha\frac{\beta\mathbb{E}(\theta)}{q^*}}\right)\bar{a}^B \tag{10}$$

The price in the asset market in the afternoon depends on the asset's market liquidity through the amount of resources that the borrowers bring to the asset market which, in turn, depends on the equilibrium price in the asset market as Equation (10) shows.

Assumption 2 (Scarcity) The relative amount of assets in the hands of lenders is such that

$$\frac{a_1^L}{a_1^B} \le \frac{1}{\beta}+\left(\mathbb{E}(\theta)-1\right)\pi_H.$$

Assumption 2 guarantees that the supply of the asset will be low enough in the asset market in the afternoon of the first period to have the price be equal to $\beta\mathbb{E}(\theta)\,d$, which maximizes the borrowers' valuation for the asset. For the rest of this section, I will assume Assumption 2 holds.

As the following proposition shows, under Assumption 2, the agents' valuations for the asset are increasing in the asset's market liquidity.

Proposition 4 *An increase in the asset's market liquidity*

a) increases the lenders' valuation for the asset in the morning of the first period, i.e.,

$$\frac{\partial v_1^L}{\partial \alpha} \geq 0,$$

b) increases the borrowers' value of selling the asset in the morning of the first period, i.e.,

$$\frac{\partial v_{1s}^B}{\partial \alpha} \geq 0, \text{ and}$$

c) strictly increases the borrowers' valuation of the asset in the morning of the first period under the optimal funding contract, i.e.,

$$\frac{\partial v_1^B}{\partial \alpha} > 0,$$

under Assumption 2.

As part a) of Proposition 4 shows, more liquid assets are valued more by lenders. First, lenders are more likely to sell the asset in the event of a default. Second, lenders sell the asset a higher price if they can access the asset market in the afternoon. Moreover, borrowers value the asset more in the morning of the first period when the asset is more liquid in the afternoon. On the one hand, borrowers can get more resources from the lender when liquidity is higher because the lenders' valuation for the asset is higher. This increases the borrowers' value of selling the asset in the morning of the first period as part b) of the proposition above shows. On the other hand, it is relatively less costly for borrowers to transfer the asset to the lender when liquidity is higher. This lower cost reduces the amount of the consumption good in the high state that a borrower can credibly commit to repaying a lender and decreases the amount of the loan. However, this last effect is exactly offset by the increase in the borrower's marginal valuation of consumption. The borrower keeps more consumption good in the high state, which he values more. This increase in consumption good holdings in the high state exactly compensates the borrower for the reduction in the loan amount due to his lower incentives to repay the lender in the high state. Therefore, the borrowers' valuation for the asset is increasing in the asset's market liquidity as stated in part c) in Proposition 4.

3.3.2 Illiquidity discount and collateral premium

The value of the asset for borrowers and lenders is maximized when the asset is perfectly liquid and the asset market in the afternoon is frictionless. Thus, there is a loss of value that is associated with the asset's illiquidity in the afternoon. This loss in value can be interpreted as an *illiquidity discount* and it is decreasing in α. Borrowers are the efficient holders of the asset at the end of the first period because they have access to investment opportunities in the morning of the second period, and given the unobservability of the projects' returns, the asset allows borrowers to invest in them. When the asset is not perfectly liquid, some assets remain in the hands of lenders and the asset is inefficiently allocated at the end of period 1. This misallocation implies borrowers will be able to raise less funds from the lenders, and thus invest less in their projects. This leads to a lower aggregate output in the afternoon of the second period and reduces the value of holding the asset in the morning of the first period. The illiquidity discount is given by

$$\left|\mathbb{E}\left(\theta\right)\left(d+\beta\mathbb{E}\left(\theta\right)d\right)-v_1^B\right|=\left(1-\alpha\right)\pi_L\beta\mathbb{E}\left(\theta\right)\left(\beta\mathbb{E}\left(\theta\right)-1\right)d,$$

where the first term on the left hand side is the borrowers' value when the asset is perfectly liquid in the afternoon, the price in the asset market is $\beta\mathbb{E}\left(\theta\right)d$ and v_1^B is the borrowers' valuation for the asset under the optimal contract. When the asset is perfectly liquid in the afternoon and the price is $\beta\mathbb{E}\left(\theta\right)d$, the borrower is able to invest all of his expected future resources in the morning of the first period and get a return greater than 1. This maximizes the borrowers' valuation for the asset.

The asset's value as collateral, which is given by the borrower's valuation of the extra amount a borrower can get from the lender by pledging the asset as collateral instead of selling it, also depends on the asset's liquidity. I'll refer to this extra valuation as *collateral premium*, which is given by

$$v_1^B-v_{1s}^B=\left(1-\alpha\right)\pi_H\beta\mathbb{E}\left(\theta\right)\left(\mathbb{E}\left(\theta\right)-1\right)d.$$

The borrower can only get additional resources from the lender by pledging the asset as collateral rather than selling it if transferring the asset is costly for the borrower. Since this cost is decreasing in the asset's liquidity, so is the asset's collateral premium. As I discussed above, the collateral premium is zero when the asset is perfectly liquid. The following corollary formalizes these arguments.

Corollary 3 *The illiquidity discount and the collateral premium are decreasing in the asset's liquidity* α*.*

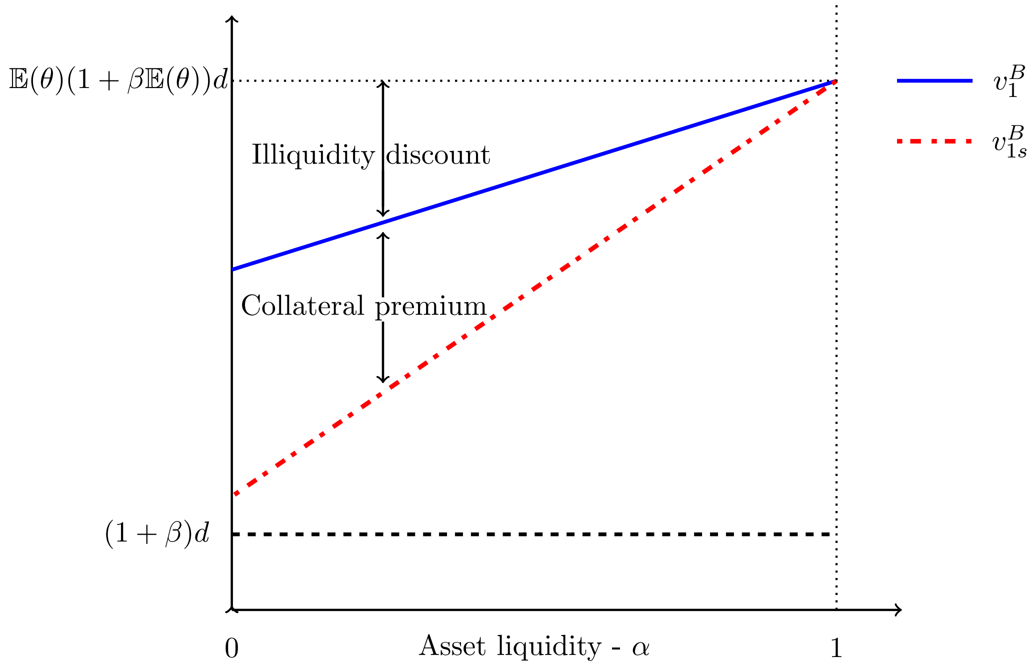

Figure 2: **Illiquidity discount and collateral premium** The blue solid line is the borrower's value under the optimal contract, the red dotted line is the borrower's value of selling the asset in the morning of the first period, the black dashed horizontal line is the fundamental value of the asset, and the black dotted horizontal line is the borrower's valuation of a perfectly liquid asset.

The proof of Corollary 3 follows directly from Proposition 4 and the definition of the borrowers' valuations. The illiquidity discount and the collateral premium are illustrated in Figure 2.

3.4 Stochastic investment opportunities

So far, I have assumed that borrowers always have access to an investment opportunity while lenders can never invest in one directly. This assumption can be relaxed to allow for probabilistic access to the risky projects. Doing so emphasizes the importance of the persistence of the investment opportunities discussed for the baseline model in Section 2.

The model presented in this section is the same as the model presented in Section 3 with the exception that now a borrower remains a borrower (and a lender remains a lender) with probability p. An agent who has a borrowing opportunity in the first periods keeps his investment opportunity in the second period with probability p while an agent without a lending opportunity in the first period acquires one in the second period with probability $(1-p)$. Whether an agent has access to an investment opportunity or not in a certain period is known at the beginning of the morning of that period. Agents who are endowed with an investment opportunity do not get any endowment of the consumption good.

The only difference between the borrowers' and the lenders' problems in this setup and those in the model in Section 3 is in the continuation values they face in the first period. In the baseline model, each agent knows which side of the market he will participate next; in the current model, their role in the economy is random. Therefore, everything derived in the main model in Section 3 holds by substituting $V_2^L(a)$ by

$$\widetilde{V}_2^L(a) = pV_2^L(a) + (1-p)\,V_2^B(a)$$

and $V_2^B(a)$ by

$$\widetilde{V}_2^B(a) = pV_2^B(a) + (1-p)\,V_2^L(a).$$

Analogous to the main model in Section 3, a borrower will only choose to transfer the asset in the high state if the lender values the asset at least as much as he does. From Proposition 3, a necessary condition for collateralized debt to be optimal is that the asset must be less than perfectly liquid. If the asset is not perfectly liquid, investment opportunities must be persistent for collateralized debt to be optimal. The following proposition formalizes this argument.

Proposition 5 *When the asset is not perfectly liquid, i.e., $\alpha \in [0,1)$, the borrower strictly prefers to use the asset as collateral over asset sales if and only if investment opportunities are persistent, i.e., if $p > 0.5$.*

Since having the asset in the morning of the second period is more valuable while having access to an investment opportunity, a borrower will value the asset more than a lender only if his likelihood of being a borrower the next period is higher than the lender's. Therefore, as long as the investment opportunities are persistent, the borrower always chooses to use the asset as collateral rather than selling it.

4 Dynamic model

I extend the model in Section 3.4 to an infinite horizon model. Formulating the model recursively makes the analysis of the equilibrium in the dynamic model almost identical to the one in the previous section by considering the appropriate continuation values. For completeness, I'll describe the full dynamic model and reference the analysis in the two-period model when necessary to avoid being repetitive.

4.1 Environment

Time is discrete and goes on forever. Each period t is divided into two subperiods, morning and afternoon. There is one storable consumption good and one durable riskless asset that pays dividends in the afternoon of each period. As in the baseline model, a units of the asset pay da units of consumption good as dividend each period. The asset is in fixed supply $\bar{A}$. There is also a continuum of risk-neutral agents. Each morning each agent can be of one of two types: a lender or a borrower. Lenders are endowed with $\bar{e}$ units of the consumption good. Borrowers have no consumption good endowment. However, each borrower i has access to a risky, constant return to scale project. One unit of consumption good invested by borrower i in the morning of period t pays a random return $\theta_t^i \in \{0, \theta_H\}$ in the afternoon of period t. The returns of the projects are identically and independently distributed across borrowers and time with $\mathbb{E}[\theta_t] > 1$. As in the previous models, the main friction in the paper is that the returns of the projects are not observable. Each morning, agents are hit by an idiosyncratic shock that determines their type. With probability p an agent is the same type he was the previous period and with probability $1-p$ he switches types, where $p > 0.5$ and $\rho \equiv 2p - 1 > 0$ is the autocorrelation of agent types in the economy.

In the morning of each period, after the idiosyncratic type shock is realized, borrowers and lenders are matched randomly. Each pair then enters into a bilateral, short-term, funding contract like the one described in Section 2 in which the borrower has all bargaining power. The terms of the contract between a borrower and a lender are determined in the morning of the period and they are settled in the afternoon of the same period. As in Section 3, the financial asset is not perfectly illiquid. Each afternoon, a random fraction $\alpha \in [0, 1]$ of the agents can access a competitive asset market.

I will assume that Assumption 1 in Section 2 holds. This assumption implies that the agents will always choose to consume consumption goods rather than saving them. Therefore, I can ignore the agents' saving decisions when characterizing the equilibrium.

Finally, there is no aggregate uncertainty in the model. All risk is idiosyncratic. However, since asset transfers between borrowers and lenders may occur over time, the aggregate state of the economy is given by the total amount of assets in the hands of borrowers and lenders. To

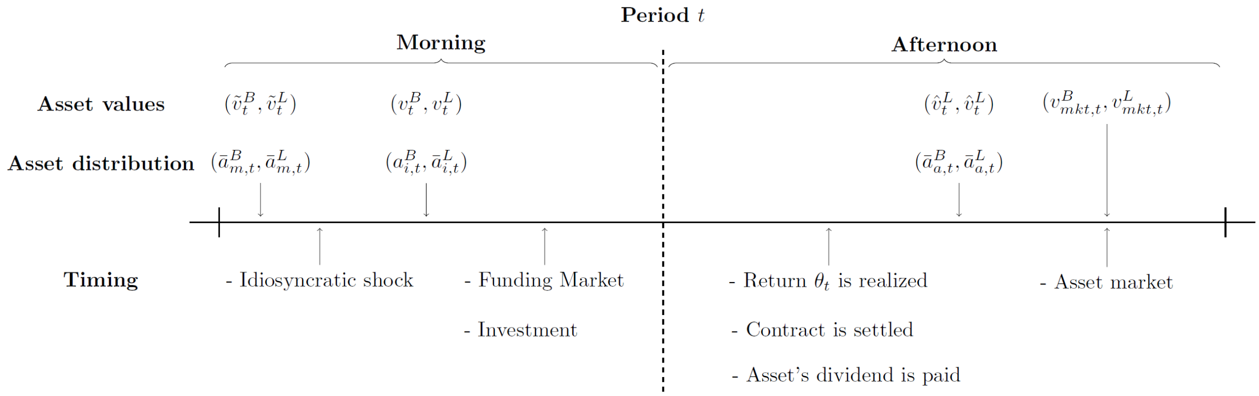

Figure 3: **Timing, asset values and asset distribution**. The figure shows the timing of the dynamic model, the value of the asset for borrowers and lenders and the asset distribution among borrowers and lenders at different stages throughout the period.

simplify the analysis it is helpful to think of the distribution of assets in the hands of borrowers and lenders in three different stages within a period: in the mornings before the idiosyncratic shocks are realized, $\left(\bar{a}_{m,t}^B, \bar{a}_{m,t}^L\right)$; in the morning after the idiosyncratic shocks have been realized, $\left(\bar{a}_{i,t}^B, \bar{a}_{i,t}^B\right)$; and in the afternoon after the funding contracts are settled and before the asset market opens, $\left(\bar{a}_{a,t}^B, \bar{a}_{a,t}^L\right)$. Figure 3 shows the timing of the dynamic model and the value and distribution of the asset among borrowers and lenders at different points within a period.

4.2 Stationary linear equilibrium

To characterize the equilibrium, I will look at equilibria in which the value function is linear in asset holdings. The shock structure together with the constant returns to scale of the technology and the linearity of the value functions implies that there exists a stationary equilibrium in which the distribution of assets between borrowers and lenders remains constant. In turn, this implies that the price in the asset market is constant over time. In the analysis below, I will focus on such stationary equilibria and ignore time subscripts and the dependence on the stationary distribution. I will guess that there is a stationary distribution and that the value functions are linear and then verify that the proposed value functions are indeed an equilibrium and that a stationary distribution of assets exists. I start by characterizing the equilibrium in the asset market in the afternoon and then look at the equilibrium in the funding market in the morning. Since the recursive formulation can be seen as a special case of the two-period model, I will refer to the analysis in Section 3.1 to avoid repetition. All omitted proofs are in the Appendix.

4.2.1 Asset market

The borrowers' and lenders' demands for the asset in the afternoon are analogous to the ones described in Section 3.1 in Equation (2) and Equation (3) with the main difference that the continuation values for borrowers and lenders take into account that the agents are subject to idiosyncratic type shocks. Let $\widetilde{v}^L$ and $\widetilde{v}^B$ be the expected marginal value of the asset for a lender and a borrower, respectively, in the morning before the idiosyncratic shocks are realized. Then,

$$\begin{aligned} \widetilde{v}^L &= pv^L + (1-p)\, v^B, \text{ and} \\ \widetilde{v}^B &= pv^B + (1-p)\, v^L, \end{aligned}$$

where v^L and v^B are, respectively, the marginal valuations for the asset for lenders and borrowers in the morning after the shocks are realized.

The equilibrium price in the asset market depends on which group of agents values the asset more in equilibrium. Since, by assumption, lenders are never constrained by their consumption good holdings—$\bar{e}$ is large enough—, the equilibrium price q^* is given by

$$q^* = \begin{cases} \max\left\{\beta\widetilde{v}^L, \min\left\{\frac{\alpha\bar{w}_B\left(\bar{a}_i^B\right)}{\alpha\bar{a}_a^L}, \beta\widetilde{v}^B\right\}\right\} & \text{if} \quad \widetilde{v}^B \geq \widetilde{v}^L \\ \beta\widetilde{v}^L & \text{if} \quad \widetilde{v}^B < \widetilde{v}^L \end{cases}, \tag{11}$$

where $\bar{w}_B$ is the total amount of consumption good in the hands of borrowers and $\bar{a}_a^L$ is the total amount of assets in the hands of borrowers before they enter the asset market in the afternoon. The total wealth in the hands of borrowers $\bar{w}_B\left(\bar{a}_i^B\right)$ is given by

$$\bar{w}_B\left(\bar{a}_i^B\right) = \pi_H\left(\theta_H k^*\left(\bar{a}_i^B\right) + d\left(\bar{a}_i^B - t_H^*\left(\bar{a}_i^B\right)\right) - r_H^*\left(\bar{a}_i^B\right)\right) + \pi_L\left(d\left(\bar{a}_i^B - t_L^*\left(\bar{a}_i^B\right)\right) - r_L^*\left(\bar{a}_i^B\right)\right), \tag{12}$$

where $k^*(a)$, $t_H^*(a)$, $t_L^*(a)$, $r_H^*(a)$ and $r_L^*(a)$ are the loan amount and the repayments in terms of asset and consumption good under the optimal funding contract for a borrower with a units of the asset. Looking at Equation (11) and Equation (12) one can see that the equilibrium price in the asset market in the afternoon only depends on the ratio of resources in the hands of borrowers and lenders, $\frac{\bar{w}_B}{\bar{a}_a^L}$, which, in turn, depends on the distribution of assets across borrowers and lenders. Therefore, if the amount of the asset in the hands of borrowers and in the hands of lenders lenders is constant, the price in the asset market is also constant across time.

Given the equilibrium price in Equation (11), the value of entering the asset market for a borrower with a units of the asset and wealth w when the price is q^* is

$$V_{mkt}^{B}(a,w) = \begin{cases} q^* a + w & \text{if } \widetilde{v}^B < \widetilde{v}^L \\ \beta \widetilde{V}^B\left(a + \frac{w}{q^*}\right) & \text{if } \widetilde{v}^B \geq \widetilde{v}^L \end{cases},$$

which can be rewritten as

$$V_{mkt}^{B}(a,w) = v_{mkt}^{B}\left(a + \frac{w}{q^*}\right),$$

where $v_{mkt}^{B} = \max\left\{\beta\widetilde{v}^B, q^*\right\}$ and $\widetilde{V}^B(a) = \widetilde{v}^B a$. The value for a borrower of entering the asset market depends on whether he is a buyer or a seller. If the borrower is a seller, i.e., if $\widetilde{v}^B < \widetilde{v}^L$, holding the asset allows him to consume q^* per unit of asset held. If the borrower is a buyer, i.e., if $\widetilde{v}^B \geq \widetilde{v}^L$, he acquires $\frac{w}{q^*}$ units of the asset in the asset market and starts the next period with $a + \frac{w}{q^*}$ assets.

Similarly, the value of entering the asset market for a lender with a units of the asset and wealth w when the price is q^* is

$$V_{mkt}^{L}(a,w) = q^* a + w.$$

The lender's value of accessing the asset market is independent of who is a buyer or a seller since whenever lenders are buyers, the price is equal to $\beta\widetilde{v}^L$.

4.2.2 Funding Market

The borrowers' problem in the funding market in the morning is analogous to the one presented in Section 3.2. A borrower's expected value of holding a assets and wealth w at the time of his funding contract's settlement is

$$\beta \hat{V}_2^B(a,w) = (1-\alpha)\left(w + \beta\widetilde{V}^B(a)\right) + \alpha V_{mkt}^{B}(a,w). \tag{13}$$

With probability $(1-\alpha)$ the borrower cannot access the asset market in the afternoon, consumes all his consumption good holdings and cannot acquire more assets to bring to the morning of the following period. With probability α the borrower can access the asset market in the afternoon and uses all of his consumption goods to buy assets, increasing the amount of assets with which he enters the following period's morning.

The lender's expected value of holding a units of the asset at the time of settlement is

$$\beta \hat{V}_2^L(a) = (1-\alpha)\beta\widetilde{V}^L(a) + \alpha q^* a, \tag{14}$$

Equations (13) and (14) are analogous to Equations (5) and (6) in the two period model in Section 3.2.

Let $w_i \equiv \theta_i k - r_i + d(a - t_i)$ be the borrower's wealth at the end of $t = 1$ if the return of his project was θ_i. A borrower with assets a solves the following problem in the morning

$$V^B(a) = \max_{k, r_L, r_H . t_L, t_H} \pi_H \beta \hat{V}_2^B(a - t_H, w_H) + \pi_L \beta \hat{V}_2^B(a - t_L, w_L) \tag{15}$$

subject to the contract being feasible, the incentive compatibility constraint in the high state

$$\hat{V}_2^B(a - t_H, w_H) \geq \hat{V}_2^B(a - t_L, w_L + (\theta_H - \theta_L) k), \tag{IC_H^d}$$

if lying is feasible, i.e., if $r_L \leq \theta_H k + d(a - t_L)$, the incentive compatibility constraint in the low state

$$\hat{V}_2^B(a - t_L, w_L) \geq \hat{V}_2^B(a - t_H, w_H + (\theta_L - \theta_H) k), \tag{IC_L^d}$$

if lying is feasible, i.e., if $r_H \leq \theta_L k + d(a - t_H)$, and the lender's participation constraint:

$$k \leq \pi_H(r_H + dt_H) + \pi_L(r_L + dt_L) + \beta\left(\pi_H \hat{V}^L(t_H) + \pi_L \hat{V}^L(t_L)\right). \tag{PC_L}$$

By inspecting the constraints in the borrower's problem in (15), it can be considerably simplified. First, in any equilibrium, the participation constraint for the lender PC_L will hold with equality. If it did not, the borrower could increase the loan amount and increase his expected utility without violating any of the additional constraints. As is typical in this kind of problems, the incentive compatibility constraint will bind in the high state and IC_H^d will hold with equality. Finally, to maximize the size of the loan, the repayment in terms of consumption good in the low state, r_L, will be the maximum possible, i.e., $r_L = d(a - t_L)$. The following proposition formalizes these arguments.

Proposition 6 *In the optimal contract, constraints PC_L, IC_H and $r_L \leq d(a - t_L)$ hold with equality.*

Using the results in Proposition 6, the borrower's problem in the funding market reduces to choosing only the contingent asset transfers. I'll proceed assuming that the incentive compatibility constraint in the low state does not bind and then verify that it is indeed the case in equilibrium. The simplified borrower's problem is

$$\begin{aligned} v^B a \quad = \quad & \max_{(t_H, t_L) \in [0, a_B]^2} \left((1-\alpha) + \alpha \frac{v_{mkt}^B}{q^*}\right) \mathbb{E}(\theta) \left(da + \beta \hat{v}^L (\pi_H t_H + \pi_L t_L)\right) \\ & -\pi_H \beta \left((1-\alpha) \widetilde{v}^B + \alpha \frac{v_{mkt}^B}{\beta}\right) \mathbb{E}(\theta)(t_H - t_L) + \beta \left((1-\alpha) \widetilde{v}^B + \alpha \frac{v_{mkt}^B}{\beta}\right)(a - t_L). \end{aligned} \tag{16}$$

subject to

$$k^* = \frac{1}{1-\theta_L}\left(da - \pi_H\psi^B\left(t_H - t_L\right) + \beta\hat{v}^L\left(\pi_H t_H + \pi_L t_L\right)\right) \quad (17)$$

$$k^* \geq \max\left\{dt_H + \pi_L\psi^B\left(t_H - t_L\right) + \beta\hat{v}^L\left(\pi_H t_H + \pi_L t_L\right), 0\right\}. \quad (18)$$

where ψ^B is the marginal rate of substitution between the asset and the consumption good for a borrower. The constraints in the problem above represent the feasibility constraints on the loan amount k and the repayment in terms of consumption goods in the high state, r_H. The marginal utility of a unit of consumption good at the time of the contract's settlement is given by $(1-\alpha) + \alpha\frac{v^B_{mkt}}{q^*}$. If the borrower cannot access the asset market, which occurs with probability $(1-\alpha)$, he consumes the goods and derives a marginal utility of one. If the borrower can access the asset market, which happens with probability α, he can use each unit of consumption good to buy $\frac{1}{q^*}$ units of the assets in the market which he values in v^B_{mkt}. Similarly, the marginal utility of the asset for the borrower is $(1-\alpha)\beta\widetilde{v}^B + \alpha v^B_{mkt}$. With probability α the borrower can access the market and gets utility v^B_{mkt} from holding the asset and with probability $(1-\alpha)$ the borrower cannot access the asset market in the afternoon and gets utility $\beta\widetilde{v}^B$. Then, the marginal rate of substitution between the asset and the consumption good for the borrower at the time of the contract's settlement is given by

$$\psi^B = \frac{(1-\alpha)\beta\widetilde{v}^B + \alpha v^B_{mkt}}{(1-\alpha) + \alpha\frac{v^B_{mkt}}{q^*}}.$$

The analogous marginal rate of substitution for the lender is $\psi^L = \beta\hat{v}^L$.

The derivative of the objective function in (16) with respect to t_H is

$$\mathbb{E}(\theta)\,\pi_H\left((1-\alpha) + \alpha\frac{\beta\widetilde{v}_B}{q^*}\right)\left(\psi^L - \psi^B\right) \quad (19)$$

and with respect to t_L is

$$\left((1-\alpha) + \alpha\frac{\beta\widetilde{v}_B}{q^*}\right)\left(\mathbb{E}(\theta)\left(\pi_L\psi^L + \pi_H\psi^B\right) - \psi^B\right). \quad (20)$$

As one can see from the expression in (19), the borrower will choose not to transfer the asset in the high state as long as he values the asset relatively more than the lender. When the borrower's marginal rate of substitution between assets and consumption goods, ψ^B, is higher than that of the lender, ψ^L, it is cheaper for the borrower to repay the lender in consumption goods than to do so in assets. Therefore, when the borrower has additional resources available, i.e., when his project is successful, he will not transfer assets to the lender. However, in the low state, the asset transfer can be positive even if the borrower values the asset relatively more than the lender since increasing t_L increases the overall transfer of value to the lender in the low state and, thus, can increases the loan quantity k^*.

The following lemma characterizes the optimal asset transfers as a function of the asset's market liquidity and the differences in valuations between borrowers and lenders.

Lemma 1 *In an affine equilibrium, a borrower will choose*

$$t_L^* = a, \text{ and}$$

$$t_H^* = \begin{cases} a & \text{if } (1-\alpha)(\widetilde{v}_L - \widetilde{v}_B) > 0 \\ x \text{ for some } x \in [0,a] & \text{if } (1-\alpha)(\widetilde{v}_L - \widetilde{v}_B) = 0 \\ 0 & \text{if } (1-\alpha)(\widetilde{v}_L - \widetilde{v}_B) < 0 \end{cases}.$$

If the asset is perfectly liquid in the afternoon, i.e., if $\alpha = 1$, the marginal rate of substitutions of borrowers and lenders are equalized and the borrower is indifferent between transferring assets or consumption goods to the lender, regardless of who values the asset more the following morning.

If the asset is less than perfectly liquid, whether the borrower values the asset more than the lender the following morning matters. If the borrower assigns the same value to the asset as the lender, he will set $t_L = t_H = a$ and "sell" the asset to the lender. In exchange, the lender will lend the borrower $\widetilde{v}_L = \widetilde{v}_B$, which is fair compensation from the borrower's perspective.

If the borrower assigns a higher value to the asset than the lender, he would be getting less than his valuation if he chose to sell the asset to the lender: the lender would still pay a price $\widetilde{v}_L$ per unit where now $\widetilde{v}_L < \widetilde{v}_B$. In this case, the borrower will only transfer the asset if he cannot avoid it, i.e., if the low return is realized and he does not have enough resources to compensate the lender in consumption goods. Transferring the asset is costly for the borrower since he gets $\widetilde{v}_L$ units for it while he values it $\widetilde{v}_B > \widetilde{v}_L$. This difference in valuation can be interpreted as the "punishment" for not being truthful, which induces truth telling and makes the state contingent contract incentive compatible. The following proposition characterizes the optimal contract in equilibrium.

Proposition 7 *In a stationary equilibrium a borrower*

a) is indifferent between using the asset as collateral and selling it to raise funds if and only if the asset is perfectly liquid in the afternoon, i.e., $\alpha = 1$, *and*

b) he strictly prefers to use the asset as collateral rather than selling it to raise funds if and only if the asset is less than perfectly liquid in the afternoon, i.e., $\alpha \in [0,1)$.

Proposition 7 is analogous to Proposition 3 and establishes the optimality of collateralized debt in dynamic settings. Current borrowers value the asset more than lenders at the funding stage in equilibrium since it allows them to invest in their (expectedly) profitable projects. Since $\rho > 0$, this implies that borrowers also value the asset more than lenders at the settlement stage only when the asset is less than perfectly liquid. Therefore, as discussed above, borrowers choose collateralized debt over asset sales to fund their investment whenever $\alpha \in [0, 1)$.

4.2.3 Asset distribution

As I mentioned above, the distribution of assets owned by borrowers and lenders is the aggregate state of the economy. Recall the total amount of assets in the hands of borrowers (lenders) is $\bar{a}_m^B$ $\left(\bar{a}_m^L\right)$ at the beginning of the morning before the idiosyncratic shocks are realized, $\bar{a}_i^B$ $\left(\bar{a}_i^L\right)$ in the morning after the idiosyncratic shocks have been realized, and $\bar{a}_a^B$ $\left(\bar{a}_a^L\right)$ in the afternoon after contracts are settled and before the asset market opens.

In a stationary equilibrium, $\widetilde{v}_B > \widetilde{v}_L$. In $\alpha < 1$, a borrower only transfers the asset to the lender if his project is unsuccessful. Also, borrowers who can access the asset market buy all the assets available in the market from the lender. Since the asset is in fixed supply and the asset has to be held by someone in equilibrium,

$$\bar{a}_{s,t}^L + \bar{a}_{s,t}^B = \bar{A} \text{ for all } s = m, i, a \text{ and all } t.$$

Therefore, the law of motion of the distribution of capital in hands of borrowers and lenders is

$$\bar{a}_{i,t}^B = p\bar{a}_{m,t}^B + (1-p)\left(\bar{A} - \bar{a}_{m,t}^B\right), \tag{21}$$

$$\bar{a}_{a,t}^B = \pi_H \bar{a}_{i,t}^B, \text{ and} \tag{22}$$

$$\bar{a}_{m,t+1}^B = \bar{a}_{a,t}^B + \alpha\left(\bar{A} - \bar{a}_{a,t}^B\right). \tag{23}$$

As Equation (21) shows, each period t only a fraction p of the assets in the hands of borrowers at the beginning of the morning remains in the hands of borrowers as $1-p$ borrowers become lenders after the idiosyncratic shocks are realized. Similarly, a fraction $1-p$ of the assets held by lenders become owned by borrowers after the idiosyncratic shocks are realized. After the funding contracts are settled in the afternoon, only the fraction π_H of borrowers who got high realized returns gets to keep their assets. This is captured by Equation (22). Finally, as Equation (23) shows, borrowers buy all the assets from the fraction α of lenders who access the asset market in the afternoon.

In a stationary equilibrium

$$\bar{a}^B_{s,t+1} = \bar{a}^B_{s,t}, \text{ for all } s = m, i, a \text{ and all } t. \tag{24}$$

Therefore, using Equations (22) – (23) and Equation (24),

$$\bar{a}^B_a = \frac{(1-\alpha)(1-p)\pi_H + \alpha}{1-(1-\alpha)(2p-1)\pi_H}\bar{A} \text{ and } \bar{a}^L_a = \frac{(1-\alpha)(1-p\pi_H)}{1-(1-\alpha)(2p-1)\pi_H}\bar{A}\ . \tag{25}$$

Given the non-observability of the projects' returns, capital is more productive in the hands of borrower. Therefore, the efficient allocation of resources in this economy puts all the asset in the hands of borrowers at the beginning of the period because this maximizes the overall production in the economy. For this to be the case, all assets in the hands of lenders after the funding contracts are settled should be transferred to the borrowers in the asset market. Whether this is the case or not, depends on the asset being perfectly liquid in the afternoon.

Proposition 8 *The allocative efficiency of the economy increases with the asset's market liquidity, i.e., the fraction of capital in the hands of borrowers is increasing α.*

In a stationary equilibrium, the relative share of assets held by lenders in the afternoon is

$$\frac{\bar{a}^L_a}{\bar{a}^B_a} = \frac{(1-\alpha)(1-p\pi_H)}{(1-\alpha)(1-p)\pi_H + \alpha},$$

which is decreasing in α. When the assets market liquidity is larger, a larger fraction of lenders can sell their assets in the asset market in the afternoon and borrowers start the following period with a larger fraction of the asset's supply, which increases the economy's allocative efficiency.

4.2.4 Valuations for the asset

To simplify the analysis, analogous to Section 3, I will focus on parameterizations such that $q^* = \beta\tilde{v}^B$ and the borrower's marginal utility of consumption is one.[8] In this case, the asset valuations in a stationary equilibrium are given by

$$v^B = \mathbb{E}(\theta)\left(v^L + (1-\alpha)\pi_H\beta\rho\left(v^B - v^L\right)\right) \tag{26}$$

$$v^L = d + \beta\left(pv^B + (1-p)v^L\right) + \beta\alpha\rho\left(v^B - v^L\right) \tag{27}$$

[8] As I show in the Appendix, the system that characterizes the valuations for the asset and the asset price in equilibrium is non-linear whenever $q^* \neq \beta\tilde{v}$. The condition such that $q^* = \beta\tilde{v}^B$ in equilibrium is characterized in the Appendix.

A borrower's valuation of the asset in the morning depends on the expected return he can get on the loan amount he procures from the lender per unit of asset held,

$$k^*(a) = \left(v^L + (1-\alpha)\,\pi_H \beta \left(\widetilde{v}^B - \widetilde{v}^L\right)\right) a.$$

The lender is always willing to give v^L to the borrower since under the optimal contract the lender knows that in the event of default he will get the asset which he values v^L. However, under the optimal contract, the lender is willing to lend the borrower additional resources in the amount of $(1-\alpha)\,\pi_H \beta \left(\widetilde{v}^B - \widetilde{v}^L\right)$ since the lender knows that the borrower is willing to pay $(1-\alpha)\,\beta \left(\widetilde{v}^B - \widetilde{v}^L\right)$ in consumption good not to lose his asset. However, the borrower is only able to pay this amount when his project is successful, which occurs with probability π_H. As discussed above, the borrower only prefers to repay the lender in consumption good when the asset is not perfectly liquid in the afternoon. This is captured by Equation 26.

In the morning, the lender values the asset for the dividends it pays in the afternoon and for the value of holding the asset in the afternoon, which depends on the asset's liquidity. With probability α, the lender can sell the asset at a price equal to the borrowers' valuation, $\beta\widetilde{v}^B$, and with probability $(1-\alpha)$ he brings the asset to the following period and gets utility $\beta\widetilde{v}^L$. Equation 27 shows this.

Consistent with the results in Proposition 4, borrowers and lenders value more liquid assets more.

Proposition 9 *Suppose $q^* = \beta\widetilde{v}^B$. Then, in a stationary equilibrium, an increase in the asset's market liquidity:*

a) increases the lenders' valuation for the asset, i.e., $\frac{\partial v^L}{\partial \alpha} > 0$, and

b) increases the borrowers' valuation for the asset , i.e., $\frac{\partial v^B}{\partial \alpha} > 0$.

4.2.5 Illiquidity discount and collateral premium

Analogous to the two-period model, the value of the asset is maximized when the asset is perfectly liquid and the asset market in the afternoon is frictionless. In this case, the values of the asset for borrowers and lenders, respectively, are

$$v^{B*} = \mathbb{E}(\theta)\,\frac{d}{1-\beta\mathbb{E}(\theta)} \text{ and } v^{L*} = \frac{d}{1-\beta\mathbb{E}(\theta)}.$$

The illiquidity discount in the dynamic model is $\left|v^{B*} - v^B\right|$ and as the corollary below shows, it is decreasing in α.

Similarly, the difference in value that can be attained with collateralized debt relative to asset sales in a period is given by

$$(1-\alpha)\,\mathbb{E}(\theta)\,\pi_H\beta\rho\left(v^B - v^L\right).$$

This value can be thought of as the one-period collateral premium, as it represents the difference in value for the borrower between using the optimal contract in every period and selling the asset today while using the optimal contract in the future. This one-period collateral premium depends on α through two channels. First, the difference in continuation values that gives rise to the collateral premium only matters when the borrower cannot access the asset market in the afternoon with some probability. Second, the asset's market liquidity affects the value of holding the asset for borrowers and lenders. As the corollary below shows, the collateral premium is also decreasing in α.

Corollary 4 *The illiquidity discount and the one-period collateral premium are decreasing in the asset's market liquidity and are zero when the asset is perfectly liquid.*

The proof of Corollary 4 follows from Proposition 9 and the characterization of v^B and v^L in the Appendix. Corollary 4 emphasizes the importance of the asset's imperfect liquidity for collateralized debt to be preferred over asset sales.

5 Conclusion

Why are liquid financial assets used as collateral instead of being sold to raise funds? I show that when financial assets are not perfectly liquid and the roles as borrowers and lenders are persistent, a borrower chooses to collateralize liquidity to raise funds when the returns of his investments are not observable. In the investment stage, borrowers value the asset relatively more than lenders because it allows them to take advantage of their investment opportunities. When the asset is not perfectly liquid, borrowers also value the asset more than lenders at the time of the funding contract's settlement. For each unit of the asset the borrower gives the lender, the borrower receives the lender's valuation in consumption good. Since the borrower values the asset more than the lender, transferring the asset to the lender is costly for the borrower. Offering the asset as collateral allows the borrower to transfer the asset only if he does not have enough resources to repay the lender in consumption good and, therefore, the borrower prefers to collateralize his assets rather than selling them.

The difference in valuations between borrowers and lenders is an equilibrium outcome and it depends on the asset's liquidity at the time of the contract's settlement. If the asset is perfectly liquid, borrowers can use the proceeds of their investment to purchase the asset in the market and they are indifferent between collateralized debt and asset sales. If the asset is less than perfectly liquid, and a borrower cannot access the asset market in the afternoon with probability one, the borrower prefers collateralized debt over outright asset sales as collateralized debt maximizes the expected amount of assets with which the borrower starts the next period. The frictions in the asset market give rise to an illiquidity discount and a collateral premium. Borrowers value the asset more when the asset's liquidity is higher because a higher liquidity increases the amount of asset that borrowers can bring to the next period and, therefore, the size of their investment. Then, the value of less than perfectly liquid assets includes an illiquidity discount which increases with the asset's illiquidity. The collateral premium is the value of the extra amount that borrowers can get from lenders by pledging the asset as collateral instead of selling it. This additional amount depends on the difference in marginal valuations between borrowers and lenders and is also increasing in the asset's illiquidity.

Changes in collateralized debt markets, such as margins and haircuts, played an important role in the recent crises (See Jermann and Quadrini (2012), Perri and Quadrini (2014)). Having a model that characterizes these objects as equilibrium outcomes is important both from a positive and a normative point of view. In positive terms, it is interesting to see where the financial shocks come from and how they interact with the economy's fundamentals. From the normative side, policies that aim at stabilizing the cycle and preventing financial crises should take into account what drives changes in the financing conditions faced by financial intermediaries, firms, and households. This paper delivers some of the insights needed to further understand collateralized debt markets.

6 Appendix

For the dynamic model, all proofs are for the stationary linear equilibrium. In this case, the solution to the borrower's problem can be characterized either by the value function V^* or by the marginal valuation v^*. To simplify the analysis, I'll write the proofs in terms of marginal valuations. For the sake of clarity, throughout most of the Appendix, I'll explicitly write the dependence of all expressions on θ_L. However, throughout the paper I assume $\theta_L = 0$.

6.1 Proof of Proposition 2

This proposition is a special case of Proposition 10 in Section 3 when $\alpha = 0$. All proofs omitted in Section 2 are special cases of the proofs in Section 3 when $\alpha = 0$.

6.2 Proof of Proposition 3

The proof of this proposition follows directly from Lemmas 4 and 5 in this section interpreting contingent asset transfers as collateralized debt and non-contingent ones as asset sales.

The borrower's problem in (1) can be considerably simplified by inspecting the constraints. First, in any equilibrium, the participation constraint for the lender will hold with equality. If it did not, the borrower could increase the loan amount and increase his expected utility without violating any of the additional constraints. Similarly, as is usual in this kind of problems, the incentive compatibility constraint will bind in the high state. Finally, in order to maximize the size of the loan, the repayment in terms of goods and assets in the low state, r_L and t_L, will be the maximum possible, i.e., $r_L = \theta_L k$ and $t_L = a$. The Lemmas 2, 3, and 4 formalize these results.

Lemma 2 *Without loss of generality, the lender's participation constraint can be replaced by:*

$$k = \pi_H (r_H + dt_H) + \pi_L (r_L + dt_L) + \beta \left(\pi_H \hat{V}_2^L (t_H) + \pi_L \hat{V}_2^L (t_L) \right)$$

in the borrower's problem.

Proof. Since the objective function of the borrower is increasing in q, if the participation constraint does not hold with equality, the borrower can always improve on the contract by increasing the loan quantity q and still satisfy all other constraints. ■

Lemma 3 *Without loss of generality, the incentive compatibility constraint in the high state can be replaced by:*

$$(1-\alpha)\left(r_H + dt_H - (r_L + dt_L)\right) = \beta\left(V_2^B\left(a - t_H + \alpha\frac{\theta_H k - r_H + d(a-t_H)}{q^*}\right) - V_2^B\left(a - t_L + \alpha\frac{\theta_H k - r_L + d(a-t_L)}{q^*}\right)\right)$$

in the borrower's problem.

Proof. Suppose that the incentive compatibility constraint does not hold with equality, i.e.,

$$(1-\alpha)\left(r_H + dt_H - (r_L + dt_L)\right) < \beta\left(V_2^B\left(a - t_H + \alpha\frac{\theta_H k - r_H + d(a-t_H)}{q^*}\right) - V_2^B\left(a - t_L + \alpha\frac{\theta_H k - r_L + d(a-t_L)}{q^*}\right)\right).$$

Then, the borrower could increase k, r_H, and r_L and increase the objective function while still satisfying all other constraints. In particular, there exists $\varepsilon_0, \varepsilon_1 > 0$ such that:

$$\begin{aligned} & (1-\alpha)\left(r_H + dt_H - (r_L + dt_L) + (\theta_H - \theta_L)\varepsilon_0\right) \\ \leq\ & \beta\left(V_2^B\left(a - t_H + \alpha\frac{-r_H + d(a-t_H)}{q^*}\right) - V_2^B\left(a - t_L + \alpha\frac{-r_L + d(a-t_L)}{q^*}\right)\right), \end{aligned}$$

$$\begin{aligned} & (1-\alpha)\left(r_L + dt_L - (r_H + dt_H) - (\theta_H - \theta_L)\varepsilon_0\right) \\ \leq\ & \beta\left(V_2^B\left(a - t_L + \alpha\frac{-r_L + d(a-t_L)}{q^*}\right) - V_2^B\left(a - t_L + \alpha\frac{-r_H + d(a-t_H)}{q^*}\right)\right) \end{aligned}$$

and

$$k + \varepsilon_0 + \varepsilon_1 = \pi_H\left(r_H + \theta_H\varepsilon_0 + dt_H\right) + \pi_L\left(r_L + \theta_L\varepsilon_0 + dt_L\right) + \beta\left(\pi_H \hat{V}_2^L(t_H) + \pi_L \hat{V}_2^L(t_L)\right).$$

Then, the contract $\{k + \varepsilon_0 + \varepsilon_1, r_L + \theta_L\varepsilon_0, r_H + \theta_H\varepsilon_0, t_{Lj}, t_{Hj}\}$ satisfies the feasibility, participation and incentive compatibility constraints while attaining a higher asset value for the borrower.

■

In the next few pages, I will ignore the incentive compatibility constraint in the low state and then verify it is satisfied.

Lemma 4 *In the optimal contract, $r_L^* = \theta_L k^*$ and $t_L^* = a$.*

Proof. Using Lemmas 2, Lemma 3 and the linearity of V_2^B and $\hat{V}_2^L$, the participation constraint of the lender can be written as participation constraint for the lender

$$k = (r_L + dt_L) - \frac{\pi_H \beta v_2^B}{\left((1-\alpha) + \alpha\frac{\beta}{q^*}v_2^B\right)}(t_H - t_L) + \beta\hat{v}_2^L\left(\pi_H t_H + \pi_L t_L\right)$$

and the incentive compatibility constraint in the high state as

$$\left(-r_H - dt_H + r_L + dt_L\right)\left((1-\alpha) + \alpha\frac{\beta}{q^*}v_2^B\right) = \beta v_2^B\left(t_H - t_L\right).$$

Using these expressions, the borrower's problem can be written as

$$\begin{aligned}&\max_{r_L,t_H,t_L}\left((1-\alpha)+\alpha\frac{\beta}{q^*}v_2^B\right)\left(\left(\mathbb{E}\left(\theta\right)-1\right)\left(r_L+dt_L\right)+da+\mathbb{E}\left(\theta\right)\beta\hat{v}_2^L\left(\pi_H t_H+\pi_L t_L\right)\right)\\&+\beta v_2^B\left(a-\pi_H t_H-\pi_L t_L\right)-\left(\mathbb{E}\left(\theta\right)-1\right)\pi_H\beta v_2^B\left(t_H-t_L\right)\end{aligned}$$

subject to the feasibility constraints. The objective function above is increasing in r_L. Therefore,

$$r_L^* = \theta_L k^* + d\left(a - t_L\right),$$

which yields

$$r_L^* + dt_L = \frac{\theta_L}{1-\theta_L}\beta\left(\pi_H\left(\hat{v}_2^L - \frac{v_2^B}{\left((1-\alpha)+\alpha\frac{\beta}{q^*}v_2^B\right)}\right)\left(t_H - t_L\right) + \hat{v}_2^L t_L\right) + \frac{da}{1-\theta_L} \tag{28}$$

and

$$k^* = \frac{1}{1-\theta_L}\left(da + \beta\hat{v}_2^L t_L + \beta\pi_H\left(\hat{v}_2^L - \frac{v_2^B}{\left((1-\alpha)+\alpha\frac{\beta}{q^*}v_2^B\right)}\right)\left(t_H - t_L\right)\right). \tag{29}$$

Using Equations (28) and (29), the borrower's problem becomes

$$\begin{aligned}&\max_{t_H,t_L}\left((1-\alpha)+\alpha\frac{\beta}{q^*}v_2^B\right)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\left(da+\beta\hat{v}_2^L\left(\pi_L t_L+\pi_H t_H\right)\right)\\&+\beta v_2^B\left(a-t_L\right)-\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\beta\pi_H v_2^B\left(t_H-t_L\right)\end{aligned} \tag{30}$$

subject to feasibility constraints. Then, the first order condition with respect to t_L

$$\begin{aligned}&\left((1-\alpha)+\alpha\frac{\beta}{q^*}v_2^B\right)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\beta\pi_L\hat{v}_2^L-\beta v_2^B\left(1-\pi_H\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\right)\\&>\ \frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\beta\left(\pi_L\hat{v}_2^L+\pi_H v_2^B\right)-\beta v_2^B>0\end{aligned}$$

since $v_2^B = \frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}d$, $\hat{v}_2^L \geq d$ and $\left((1-\alpha)+\alpha\frac{\beta}{q^*}v_2^B\right) > 1$. This implies $t_L^* = a$ and $r_L^* = \theta_L k^*$. ■

Lemma 5 *In the optimal contract, $t_H = 0$ as long as $\alpha \in [0,1)$ and If $\alpha = 1$, $t_H \in [0,a]$.*

Proof. From the borrower's problem in Equation 30 in the proof of Lemma 4 we have that the first order condition with respect to t_H is

$$\frac{\left(\mathbb{E}\left(\theta\right)-\theta_L\right)}{1-\theta_L}\left((1-\alpha)+\alpha\frac{\beta}{q^*}v_2^B\right)\beta\pi_H\left(\hat{v}_2^L - \frac{v_2^B}{\left((1-\alpha)+\alpha\frac{\beta}{q^*}v_2^B\right)}\right)$$

For all $\alpha \in [0,1)$, the first order condition is negative since:

$$\hat{v}_2^L - \frac{v_2^B}{\left((1-\alpha) + \alpha \frac{\beta}{q^*} v_2^B\right)} = \frac{\left((1-\alpha)\left(\hat{v}_2^L - v_2^B\right) + \alpha\left(\frac{\beta}{q^*}\hat{v}_2^L - 1\right) v_2^B\right)}{\left((1-\alpha) + \alpha \frac{\beta}{q^*} v_2^B\right)} < 0$$

since $\hat{v}_2^L - v_2^B < 0$ and $\frac{\beta}{q^*}\hat{v}_2^L \leq 1$. Therefore, the borrowers chooses $t_H^* = 0$.

If $\alpha = 1$, the first order condition becomes

$$\frac{(\mathbb{E}(\theta) - \theta_L)}{1-\theta_L} \frac{\beta}{q^*} v_2^B \beta \pi_H \left(\hat{v}_2^L - \frac{q^*}{\beta}\right) = 0$$

since $\hat{v}_2^L = (1-\alpha) d + \alpha \frac{q^*}{\beta}$. In this case, the borrower is indifferent between any value of $t_H^* \in [0, a]$. Note that since $r_L = \theta_L k$ and $r_H > r_L$, not reporting the truth in the low state is not feasible and the incentive compatibility constraint in the low state can be ignored. ■

Proposition 10 *Choosing a funding contract at $t = 1$ where*

$$\begin{aligned}
k_1^* &= \frac{a}{1-\theta_L}\left(d + \beta \hat{v}_2^L + \beta \pi_H \left(\frac{v_2^B}{\left((1-\alpha) + \alpha \frac{\beta v_2^B}{q^*}\right)} - \hat{v}_2^L\right)\right) \\
r_{1L}^* &= \theta_L k_1^*,\ r_{1H}^* = \theta_L k_1^* + da + \frac{\beta v_2^B a}{\left((1-\alpha) + \alpha \frac{\beta v_2^B}{q^*}\right)} \\
t_{1L}^* &= a, \text{ and } t_{1H}^* = 0.
\end{aligned}$$

is optimal for the borrower. This is the unique optimal contract if $\alpha \in [0,1)$.

Proof. The proof follows directly from Lemmas 4 and 5 in this subsection. ■

6.3 Proof of Proposition 4

Given assumption 2, $q^* = \beta \mathbb{E}(\theta) d$. Then,

$$\begin{aligned}
v_1^L &= d + \beta d + \alpha \pi_H \beta (\mathbb{E}(\theta) - 1) d, \\
v_{1s}^B &= \mathbb{E}(\theta) v_1^L, \text{ and} \\
v_1^B &= d + \beta \mathbb{E}(\theta) d - \pi_L (1-\alpha)(\mathbb{E}(\theta) - 1) \beta.
\end{aligned}$$

Differentiating this expressions with respect to α gives the results in the proposition.

6.4 Proof of Proposition 5

As long as the borrower values the asset more than lender at the time of settlement in period 1, the borrower will find it costlier to transfer the asset to lender and, thus, strictly prefer to use the asset as collateral. This happens when:

$$(1-\alpha)\widetilde{v}_2^L + \alpha\frac{q^*}{\beta} - \frac{\widetilde{v}_2^B}{\left((1-\alpha)+\alpha\frac{\beta}{q^*}\widetilde{v}_2^B\right)} < 0$$

which is the same as

$$(1-\alpha)\left((1-\alpha)\left(\widetilde{v}_2^L-\widetilde{v}_2^B\right)+\alpha\frac{\beta}{q^*}\widetilde{v}_2^B\left(\widetilde{v}_2^L-\frac{q^*}{\beta}\right)+\alpha\left(\frac{q^*}{\beta}-\widetilde{v}_2^B\right)\right)<0.$$

This condition can only be satisfied if $\alpha < 1$. Since q^* is between $\beta\widetilde{v}_2^L$ and $\beta\widetilde{v}_2^B$, a necessary condition for the inequality above to hold is $\widetilde{v}_2^L - \widetilde{v}_2^B < 0$ which is the same as

$$(2p-1)\left(v_2^B - v_2^L\right) > 0,$$

which will be satisfied iff $0.5 < p$, since $v_2^B - v_2^L > 0$, as long as $\alpha \in [0,1)$.

6.5 Dynamic Asset market

All proofs are for the linear stationary equilibrium. In this case, the solution to the borrower's problem can be characterized either by the value function V^* or by the marginal valuation v^*. To simplify the analysis, I'll write the proofs terms of marginal valuations.

Given the linear structure of the borrowers' and lenders' demand in the asset market, the equilibrium price in the asset market depends on which group of agents values the asset more in equilibrium. The equilibrium price is given by

$$q^* = \begin{cases} \max\left\{\beta\widetilde{v}^L, \min\left\{\frac{\alpha\bar{w}_B}{\alpha\bar{a}_L^a}, \beta\widetilde{v}^B\right\}\right\} & \text{if } \widetilde{v}^B > \widetilde{v}^L \\ \beta\widetilde{v}^L & \text{if } \widetilde{v}^B = \widetilde{v}^L \\ \beta\widetilde{v}^L & \text{if } \widetilde{v}^B < \widetilde{v}^L \end{cases},$$

since, by assumption, $\bar{e}$ is large enough such that lenders always have enough resources in the funding and asset markets. Therefore, $q^* \geq \beta\widetilde{v}^L$.

The value of a borrower of entering the asset market when the price is q^* is

$$V_{mkt}^B(a,w) = \begin{cases} q^*a + w & \text{if } \widetilde{v}^B < \widetilde{v}^L \\ \beta\widetilde{v}^B a + w & \text{if } \widetilde{v}^B = \widetilde{v}^L \\ \beta\widetilde{V}^B\left(a+\frac{w}{q^*}\right) & \text{if } \widetilde{v}^B > \widetilde{v}^L \end{cases}.$$

Given the equilibrium price,

$$V_{mkt}^{B}(a,w)=\begin{cases} q^{*}a+w & \text{if } \widetilde{v}^{B}<\widetilde{v}^{L} \\ \beta\widetilde{V}^{B}\left(a+\frac{w}{q^{*}}\right) & \text{if } \widetilde{v}^{B}\geq\widetilde{v}^{L} \end{cases}.$$

Note that

$$V_{mkt}^{B}(a,w)=v_{mkt}^{B}\left(a+\frac{w}{q^{*}}\right),$$

where

$$v_{mkt}^{B}=\max\left\{\beta\widetilde{v}^{B},q^{*}\right\}$$

is the marginal utility of the asset for the borrower when he enters the asset market. The maximum operator takes into account the possibility of the borrower being a buyer or a seller in equilibrium.

Analogously, the value of a lender of entering the asset market when the price is q^{*} is

$$V_{mkt}^{L}(a,w)=\begin{cases} q^{*}a+w & \text{if } \widetilde{v}^{B}>\widetilde{v}^{L} \\ \beta\widetilde{v}^{L}a+w & \text{if } \widetilde{v}^{B}=\widetilde{v}^{L} \\ \beta\widetilde{V}^{L}\left(a+\frac{w}{q^{*}}\right) & \text{if } \widetilde{v}^{B}<\widetilde{v}^{L} \end{cases},$$

which, given the equilibrium price above, becomes

$$V_{mkt}^{L}(a,w)=q^{*}a+w.$$

6.6 Proof of Proposition 6

The proof of follows from Lemmas 6, 7 and 8 below.

Lemma 6 *Without loss of generality, PC_{L} can be replaced by*

$$k=\pi_{H}(r_{H}+dt_{H})+\pi_{L}(r_{L}+dt_{L})+\beta\left(\pi_{H}\hat{v}^{L}t_{H}+\pi_{L}\hat{v}^{L}t_{L}\right), \tag{PC_L'}$$

in the borrower's problem.

Proof. Let v^{*} be the solution to the borrower's problem in a stationary equilibrium. Let $\{v_{j}\}_{j\geq0}$ be such that $\lim_{j\to\infty}v_{j}=v^{*}$, where:

$$\begin{aligned} v_{j}a \quad = \quad & \left((1-\alpha)+\alpha\frac{v_{mkt}^{B}}{q^{*}}\right)(\pi_{H}w_{Hj}+\pi_{L}w_{Lj}) \\ & +\left((1-\alpha)\beta\widetilde{v}_{j+1}^{B}+\alpha v_{mkt}^{B}\right)(a-\pi_{H}t_{Hj}-\pi_{L}t_{Lj}) \end{aligned} \tag{31}$$

for some feasible and incentive compatible $\{k_j, r_{Lj}, r_{Hj}, t_{Lj}, t_{Hj}\}$ that satisfies the participation constraint PC, where $w_{ij} \equiv \theta_i k_j - r_{ij} + d(a - t_{ij})$,

$$\widetilde{v}_{j+1}^B = pv_{j+1}^B + (1-p)\, v_{j+1}^L$$

and q_j^* is the equilibrium price in the asset market when the borrowers' and lenders' value of holding the asset the following morning are v_{j+1}^B and v_{j+1}^L, respectively. The maximum operator in the expression above takes into account the possibility of the borrower being a seller in the asset market in equilibrium.

Suppose that for some $j \geq 0$, $(k_j, r_{Lj}, r_{Hj}, t_{Lj}, t_{Hj})$ is such that PC is slack. Then, one could increase k_j and increase v_j to v_j^0 while still satisfying all the other constraints. Let $\left\{v_j'\right\}_{j\geq 0}$ be a sequence identical to $\{v_j\}_{j\geq 0}$ if at $(k_j, r_{Lj}, r_{Hj}, t_{Lj}, t_{Hj})$ PC holds with equality and v_j^0 otherwise. Then, by construction, $v_j^{'} \geq v_j$ and therefore,

$$\lim_{j\to\infty} v_j' \geq \lim_{j\to\infty} v_j = v^*.$$

Therefore, one can replace PC_L by Equation $PC_L^{'}$ in the borrower's problem. ■

Lemma 7 *Without loss of generality, the incentive compatibility constraint in the high state can be replaced by:*

$$\left((1-\alpha) + \alpha \frac{v_{mkt}^B}{q^*}\right)(r_H - r_L + d(t_H - t_L)) = \beta \left((1-\alpha)\,\widetilde{v}^B + \alpha \frac{v_{mkt}^B}{\beta}\right)(t_H - t_L) \qquad (IC_H')$$

in the borrower's problem.

Proof. Let v^* be the solution to the borrower's problem. Let $\{v_j\}_{j\geq 0}$ be such that $\lim_{j\to\infty} v_j = v^*$, where v_j is given by Equation (31). Suppose that for some $s \geq 0$, no incentive compatibility constraint binds. Then, there exists $\varepsilon_s > 0$, such that the incentive compatibility constraint IC_H^d is slack

$$\left((1-\alpha) + \alpha \frac{v_{mkt,s}^B}{q_s^*}\right)(r_{Hs} - r_{Ls} + d(t_{Hs} - t_{Ls}) + (\theta_H - \theta_L)\,\varepsilon_s) > \beta \left((1-\alpha)\,\widetilde{v}_{s+1}^B + \alpha \frac{v_{mkt,s}^B}{\beta}\right)(t_{Hs} - t_{Ls})$$

Replace $\{k_s, r_{Ls}, r_{Hs}, t_{Ls}, t_{Hs}\}$ by $\{k_s + \varepsilon_s + \varepsilon_0, r_{Ls} + \theta_L\varepsilon_s, r_{Hs} + \theta_H\varepsilon_s, t_{Lj}, t_{Hj}\}$, where $\varepsilon_0 > 0$ is such that the participation constraint binds. This contract still satisfies all the constraints, but it attains a value $v_s^0 > v_s$. Therefore, at least one IC constraint binds in equilibrium.

If $r_{Hs} + dt_{Hs} > r_{Ls} + dt_{Ls}$, IC_H^d is the only relevant incentive compatibility constraint, because lying is not feasible in the low state. Then, by the argument above, for all s such that $r_{Hs} + dt_{Hs} > r_{Ls} + dt_{Ls}$, IC_H binds.

Now consider those $s \geq 0$ such that $r_{Hs} + dt_{Hs} \leq r_{Ls} + dt_{Ls} < \theta_L k_s + da_B$. If IC_L^d binds, one could keep $r_{Hs} - r_{Ls_s}$ constant by increasing both r_{Ls} and r_{Hs} and by increasing k_s to keep the participation constraint binding, which would result in an increase in the objective function. Let this new value be v_s^0. If for $s \geq 0$, $r_{Hs} + dt_{Hs} \leq r_{Ls} + dt_{Ls} = \theta_L k_s + da_B$, IC_L^d does not bind unless IC_H^d binds. Suppose IC_L^d binds and IC_H^d does not. Then, one could increase r_{Hs} while still satisfying incentive compatibility and relaxing the participation constraint. Therefore, one could increase k_s, which would increase the objective function and give a value $v_s^0 \geq 0$.

Therefore, if IC_H^d was not binding for some $s \geq 0$, one could construct a new sequence $\left\{v_j'\right\}_{j\geq 0}$, $v_j' \geq v_j$, such that $v_j' = v_j$ if the incentive compatibility constraint in the high state binds and $v_j' = v_j^0$ if it does not. By construction,

$$\lim_{j\to\infty} v_j^{'} \geq \lim_{j\to\infty} v_j = v^*.$$

Therefore, without loss of generality one can concentrate on those sequences that are feasible in which IC_H^d holds with equality. ■

Lemma 8 *Without loss of generality, the feasibility constraints on the contingent transfers in consumption good can be replaced by:*

$$r_L = \theta_L k + d\left(a - t_L\right) \text{ and } r_H \geq 0$$

in the borrower's problem.

Proof. By assumption, $k \leq \bar{e}$ will not bind in a solution to the borrower's problem. Using Lemma 6, the participation constraint can be assumed to hold with equality, and using this in the objective function it is easy to see that the objective function is always increasing in the amount of the loan k. Using Lemma 7, the incentive compatibility constraint holds with equality, which implies that the upper bound for k is given by the maximum amount that can be repaid in the low state, i.e., by $r_L = \theta_L k + d\left(a - t_L\right)$.

Let v^* be a solution to the borrower's problem. Let $\{v_j\}$ be a sequence such that $lim_{j\to\infty} v_j = v^*$ and where v_j is given by Equation (31) and satisfies IC_H^d and PC with equality. These two equation imply

$$r_{Lj} = k_j - dt_{Lj} + \pi_H \psi_j^B \left(t_{Hj} - t_{Lj}\right) - \beta\left(\pi_H \hat{v}_{j+1}^L t_H + \pi_L \hat{v}_{j+1}^L t_L\right), \text{ and} \tag{32}$$

$$r_{Hj} = k_j - dt_{Hj} - \pi_L \psi_j^B \left(t_{Hj} - t_{Lj}\right) - \beta\left(\pi_H \hat{v}_{j+1}^L t_H + \pi_L \hat{v}_{j+1}^L t_L\right), \tag{33}$$

where

$$\psi_j^B \equiv \beta \frac{(1-\alpha)\widetilde{v}_{j+1}^B + \alpha \frac{v_{mkt,j}^B}{\beta}}{(1-\alpha) + \alpha \frac{v_{mkt,j}^B}{q_j^*}}$$

is the marginal rate of substitution between assets and consumption goods for the borrower. Then, for all j, the contract can be summarized by $\{q_j, t_{Lj}, t_{Hj}\}$. The feasibility constraints for r_L and r_H imply the following constraints on q_j:

$$k_j \leq \frac{1}{1-\theta_L}\left(da_B - \pi_H \psi_j^B + \beta\left(\pi_H \hat{v}_{j+1}^L t_H + \pi_L \hat{v}_{j+1}^L t_L\right)\right), \tag{34}$$

$$k_j \geq dt_{Lj} - \pi_H \psi_j^B + \beta\left(\pi_H \hat{v}_{j+1}^L t_H + \pi_L \hat{v}_{j+1}^L t_L\right), \tag{35}$$

$$k_j \geq \frac{1}{1-\theta_H}\left(da_B + \pi_L \psi_j^B + \beta\left(\pi_H \hat{v}_{j+1}^L t_H + \pi_L \hat{v}_{j+1}^L t_L\right)\right), \tag{36}$$

$$k_j \geq dt_{Hj} + \pi_L \psi_j^B + \beta\left(\pi_H \hat{v}_{j+1}^L t_H + \pi_L \hat{v}_{j+1}^L t_L\right). \tag{37}$$

Construct the following sequence $\left\{v_j'\right\}$: if $\{k_j, t_{Lj}, t_{Hj}\}$ is such that Eq.(34) holds with equality, set $v_j' = v_j$. If $\{k_j, t_{Lj}, t_{Hj}\}$ is such that Eq.(34) is slack, let v_j' be the value attained by the contract that satisfies Eq.(34) with equality. Since the transfers in terms of consumption good are defined by Eq.(32) and Eq.(33), this contract is still incentive compatible and feasible. Moreover, $k_j' > k_j$ and $v_j' > v_j$. Therefore,

$$\lim_{j\to\infty} v_j' \geq \lim_{j\to\infty} v_j = v^*$$

and without loss of generality I can concentrate on the sequences $\{v_j\}$ as defined above in Eq.(31), such that the loan quantities $\{k_j\}$ satisfy Eq.(34) with equality. Having this constraint hold with equality implies that $r_{Lj} = \theta_L k_j + d(a_B - t_{Lj})$. Since $k_j \geq 0$ always, all contracts along this sequence satisfy $r_L > 0$, which is the same as satisfying Eq.(35) with strict inequality.

Suppose that for some j Eq.(36) holds with equality. This implies $r_{Hj} = \theta_H k_j + d(a_B - t_{Hj})$ and since $r_{Lj} = \theta_L k_j + d(a_B - t_{Lj})$, this implies that the participation constraint is slack, and that the producer is giving the non-producer all the gains from the project. From Lemma 6, there exists a feasible and incentive compatible contract that attains a higher value that contract j and therefore, without loss of generality one can ignore sequences in which for some elements j, Eq.(36) holds with equality. ■

6.7 Proof of Lemma 1

The proof follows from the lemmas below.

Note that the marginal utility of the asset for the borrowers and lenders satisfies

$$\begin{aligned} v^B &= \left((1-\alpha)+\alpha\frac{v^B_{mkt}}{q^*}\right)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(d+\beta\hat{v}^L\left(\pi_H\frac{\partial t_H}{\partial a}+\pi_L\frac{\partial t_L}{\partial a}\right)\right) \\ &\quad +\left((1-\alpha)\beta\widetilde{v}^B+\alpha v^B_{mkt}\right)\left(\left(1-\frac{\partial t_L}{\partial a}\right)-\tfrac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\pi_H\left(\frac{\partial t_H}{\partial a}-\frac{\partial t_L}{\partial a}\right)\right), \\ \widetilde{v}^B &= pv^B+(1-p)v^L, \\ v^L &= d+\beta\hat{v}^L, \text{ and} \\ \hat{v}^L &= \alpha\frac{q^*}{\beta}+(1-\alpha)\left(pv^L+(1-p)v^B\right). \end{aligned}$$

Lemma 9 *In equilibrium, $t_L = a$.*

Proof. Suppose $t_L < a$. Then $t_H = 0$ since Equation (20) weakly less than zero implies $\beta\hat{v}^L < M$. Suppose $\widetilde{v}^L > \widetilde{v}^B$. Then,

$$\psi^B = \beta\left((1-\alpha)\widetilde{v}^B+\alpha\widetilde{v}^L\right) < \beta\widetilde{v}^L \le \beta\hat{v}_L$$

since $q^* = \widetilde{v}^L$ which is a contradiction. Therefore, if $t_L = 0$ it must be the case that $v^B \ge v^L$. In this case,

$$\psi^B = \frac{\beta\widetilde{v}^B}{(1-\alpha)+\alpha\frac{\beta\widetilde{v}^B}{q^*}}.$$

and

$$\begin{aligned} v^B &= \left((1-\alpha)+\alpha\frac{\beta\widetilde{v}^B}{q^*}\right)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}d+\beta\widetilde{v}^B, \\ \widetilde{v}^B &= pv^B+(1-p)v^L, \\ v^L &= d+\beta\hat{v}^L, \text{ and} \\ \hat{v}^L &= \alpha\frac{q^*}{\beta}+(1-\alpha)\left(pv^L+(1-p)v^B\right). \end{aligned}$$

If $\pi_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}-1 > 0$ then the first order condition with respect to t_L is positive which is a contradiction. If $\pi_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}-1 < 0$

$$\begin{aligned} \beta\left(\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(\pi_L\left((1-\alpha)+\alpha\frac{\beta\widetilde{v}_B}{q^*}\right)\hat{v}^L+\pi_H\widetilde{v}^B\right)-\widetilde{v}^B\right) &\ge \\ \beta\frac{d}{1-\beta}\left((1-\alpha)+\alpha\frac{\beta\widetilde{v}^B}{q^*}\right)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}\left(\pi_L+\pi_H\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}-1\right) &> 0 \end{aligned}$$

since $v^B \ge v^L$ and $\rho > 0$

$$\widetilde{v}^B \le \frac{\left((1-\alpha)+\alpha\frac{\beta\widetilde{v}^B}{q^*}\right)\frac{(\mathbb{E}(\theta)-\theta_L)}{1-\theta_L}d}{1-\beta}$$

which is a contradiction. ■

Lemma 10 *sign* $\left(M-\beta\hat{v}^{L}\right)=sign\left((1-\alpha)\left(\widetilde{v}^{B}-\widetilde{v}^{L}\right)\right)$

Proof. From the definition of ψ^B and ψ^L we have

$$\begin{aligned}\psi^{B}-\psi^{L} &= \psi^{B}-\beta\hat{v}^{L}\\ &= \frac{\beta\widetilde{v}^{B}}{(1-\alpha)+\alpha\frac{\beta\widetilde{v}^{B}}{q^{*}}}-\alpha q^{*}-(1-\alpha)\beta\widetilde{v}_{L}\\ &= (1-\alpha)\left(\frac{\left(\beta\widetilde{v}^{B}-q^{*}\right)}{(1-\alpha)+\alpha\frac{\beta\widetilde{v}^{B}}{q^{*}}}-\left(\beta\widetilde{v}_{L}-q^{*}\right)\right),\end{aligned}$$

where

$$sign\left(\left(\beta\widetilde{v}^{B}-q^{*}\right)-\left((1-\alpha)+\alpha\frac{\beta\widetilde{v}^{B}}{q^{*}}\right)\left(\beta\widetilde{v}_{L}-q^{*}\right)\right)=sign\left(\widetilde{v}^{B}-\widetilde{v}^{L}\right)$$

since $q^{*}\in\left[\min\left\{\widetilde{v}^{B},\widetilde{v}^{L}\right\},\max\left\{\widetilde{v}^{B},\widetilde{v}^{L}\right\}\right]$. ■

6.8 Proof of Proposition 7

The proof follows from the lemma below and the derivative of the objective function with respect to t_H in (19).

Lemma 11 *In equilibrium,* $\beta\hat{v}^{L}\leq\psi^{B}$ *and* $\beta\hat{v}^{L}=\psi^{B}$ *if and only if* $\alpha=1$

Proof. Suppose that $\beta\hat{v}^{L}>\psi^{B}$. Then, $t_H=a_B$ and $t_L=a_B$. In this case

$$v^{B}=\left((1-\alpha)+\alpha\frac{\psi^{B}}{\beta q^{*}}\right)\frac{\left(\mathbb{E}\left(\theta\right)-\theta_{L}\right)}{1-\theta_{L}}\left(d+\beta\hat{v}^{L}\right)>v^{L}$$

since $\left((1-\alpha)+\alpha\frac{\psi^{B}}{\beta}\right)\frac{(\mathbb{E}(\theta)-\theta_{L})}{1-\theta_{L}}>1$. Then, since $\rho>0$, $\widetilde{v}^{B}>\widetilde{v}^{L}$ and

$$\psi^{B}=\frac{\beta\widetilde{v}^{B}}{(1-\alpha)+\alpha\frac{\beta\widetilde{v}^{B}}{q^{*}}}\geq\frac{\beta\widetilde{v}^{B}}{\frac{\beta\widetilde{v}^{B}}{q^{*}}}=q^{*}\geq\beta\hat{v}^{L}\tag{38}$$

which is a contradiction. Thus, $\beta\hat{v}^{L}\leq\psi^{B}$.

Suppose that $\alpha=1$, then $\beta\hat{v}^{L}=q^{*}$ and

$$\psi^{B}=\frac{v_{mkt}^{B}}{\frac{v_{mkt}^{B}}{q_{j}^{*}}}=q^{*}=\beta\hat{v}^{L}.$$

If $\beta\hat{v}^{L}=\psi^{B}$, then

$$\frac{(1-\alpha)\beta\widetilde{v}^{B}+\alpha v_{mkt}^{B}}{(1-\alpha)+\alpha\frac{v_{mkt}^{B}}{q^{*}}}=\alpha q^{*}+(1-\alpha)\beta\widetilde{v}^{L}.\tag{39}$$

Suppose that $\widetilde{v}^{L}>\widetilde{v}^{B}$. Then $v_{mkt}^{B}=q^{*}=\beta\widetilde{v}^{L}$ and Equation (39) becomes

$$(1-\alpha)\widetilde{v}^{B}+\alpha\widetilde{v}^{L}=\widetilde{v}^{L}$$

which only holds if $\alpha = 0$.

Suppose that $\widetilde{v}^L \leq \widetilde{v}^B$. Then, $v^B_{mkt} = \beta\widetilde{v}^B$ and $\beta\widetilde{v}^B \geq q^* \geq \beta\widetilde{v}^L$. In this case,

$$\psi^B = \frac{\beta\widetilde{v}^B}{(1-\alpha)+\alpha\frac{\widetilde{v}^B}{q^*}} \geq \frac{\beta\widetilde{v}^B}{\frac{\beta\widetilde{v}^B}{q^*}} = q^* \geq \beta\hat{v}^L.$$

Then, to have $\hat{v}^L = \frac{\psi^B}{\beta}$ both inequalities above must hold with equality which implies either $\alpha = 1$ or

$$\widetilde{v}^B = \frac{q^*}{\beta} = \widetilde{v}^L,$$

or both. Suppose $\widetilde{v}^L = \widetilde{v}^B$ in equilibrium, then, $q^* = \widetilde{v}^B$ and the marginal valuations must solve,

$$\begin{aligned} v^B &= \frac{\mathbb{E}(\theta) - \theta_L}{1-\theta_L} d + \beta\widetilde{v}^B, \text{ and} \\ v^L &= d + \beta\hat{v}^L. \end{aligned}$$

which imply $\widetilde{v}^L < \widetilde{v}^B$ for $\rho > 0$ since $\mathbb{E}(\theta) > 1$. ■

6.9 Dynamic equilibrium marginal valuations

The marginal valuations of the asset in equilibrium are given by the solution to the following system

$$v^B = \left((1-\alpha) + \alpha\frac{\beta\widetilde{v}^B}{q^*}\right)\mathbb{E}(\theta)\left(v^L + \pi_H\left(\frac{\beta\widetilde{v}^B}{\left((1-\alpha)+\alpha\frac{\beta\widetilde{v}^B}{q^*}\right)} - \left(\alpha q^* + (1-\alpha)\beta\widetilde{v}^L\right)\right)\right) \quad (40)$$

$$v^L = d + \alpha q^* + \beta(1-\alpha)\widetilde{v}^L \quad (41)$$

$$\widetilde{v}^B = pv^B + (1-p)v^L \quad (42)$$

$$\widetilde{v}^L = pv^L + (1-p)v^B. \quad (43)$$

and

$$q^* = \max\left\{\beta\widetilde{v}^L, \min\left\{\frac{\alpha\bar{w}_B\left(\bar{a}_i^B\right)}{\alpha\bar{a}_a^L}, \beta\widetilde{v}^B\right\}\right\} \quad (44)$$

where $\bar{a}_i^B = \frac{\bar{a}_a^B}{\pi_H}$ and

$$\bar{w}_B(a) = \left(\mathbb{E}(\theta)\left(d + \pi_H\frac{\beta\widetilde{v}^B}{(1-\alpha)+\alpha\frac{\beta\widetilde{v}^B}{q^*}} + \beta\widetilde{v}^L\pi_L\right) - \pi_H\beta\widetilde{v}^B\right)a. \quad (45)$$

The asset valuations in the system given by Equations (40)– (45) are analogous to the ones presented in the static model in Section 3. This system is non-linear in the asset valuations whenever $q^* \neq \beta\widetilde{v}^B$.

If $q^* = \beta\widetilde{v}^B$, the system becomes linear and in equilibrium

$$\begin{aligned} v^L &= \frac{d}{1-\beta\left(p-\mathbb{E}(\theta)\frac{\beta\pi_H(\alpha-1)(2p-1)+1}{1-\mathbb{E}(\theta)\beta\pi_H(1-\alpha)(2p-1)}(p-1)\right)-\alpha\beta(2p-1)\frac{\mathbb{E}(\theta)-1}{1-\mathbb{E}(\theta)\beta\pi_H(1-\alpha)(2p-1)}} \\ v^B &= \omega v^L \end{aligned}$$

where

$$\omega \equiv \frac{\mathbb{E}(\theta)\left(1-\beta\pi_H(1-\alpha)(2p-1)\right)}{1-\mathbb{E}(\theta)\beta\pi_H(1-\alpha)(2p-1)} > 1.$$

For $q^* = \beta\widetilde{v}^B$ in equilibrium it must be the case that

$$\bar{w}_B\left(\frac{a_a^B}{\pi_H}\right) \geq \beta\widetilde{v}^B\left(\bar{A}-a_a^B\right)$$

which is the same as

$$\left(\theta_H\left(d+\pi_H\beta\widetilde{v}^B+\beta\widetilde{v}^L\pi_L\right)-\beta\widetilde{v}^B\right)a_a^B \geq \beta\left(1+p(\omega-1)\right)\frac{\left(\bar{A}-a_a^B\right)}{a_a^B}v^L,$$

which, from computational examples, holds for a non-empty set of parameters.

Moreover, the one-period collateral premium

$$(1-\alpha)\beta\pi_H\rho\left(v^B-v^L\right) = \beta\rho\pi_H(1-\alpha)(\omega-1)v^L$$

is decreasing in α since

$$sign\left(\frac{\partial\left((1-\alpha)(\omega-1)v^L\right)}{\partial\alpha}\right) = sign\left(p\beta\left(\mathbb{E}(\theta)-1\right)-(1-\beta)\right)$$

and $\beta E(\theta) < 1$.

6.10 Long-term contract

I show that focusing on short-term contracts is without loss of generality because the sequence of optimal short-term contracts in the baseline model implements the long-term contract that satisfies the participation constraints for the lender in periods 1 and 2.

Definition 5 (Long-term Contract) *A long-term contract ψ consists of:*

1. *A loan amount in period* 1, k.
2. *Loan amounts in period* 2 *contingent on the report in period* 1, k_L *and* k_H.
3. *Repayments in terms of the consumption good*

   in period 1, *contingent on the report in period* 1, r_L *and* r_H

in period 2*, contingent on the reports in periods* 1 *and* 2*,* r_{LL}*,* r_{LH} *,* r_{HL}*, and* r_{HH}*.*

4. *Repayments in terms of assets*

   in period 1*, contingent on the report in period* 1*,* t_L *and* t_H

   in period 2*, contingent on the report in periods* 1 *and* 2*,* t_{LL}*,* t_{LH}*,* t_{HL}*, and* t_{HH}*.*

As in the case of the short-term contract, in equilibrium a long-term contract has to be feasible and incentive compatible. These conditions are expressed in the following borrower's problem as constraints $1-3$. Finally, I assume that the lender can walk away from the contract at any time and, therefore, his participation constraints have to be satisfied in periods 1 and 2. These are constraints 4 and 5 below. The borrower chooses a long-term contract to maximize his expected utility:

$$\begin{aligned}\max_{\psi} \mathbb{E}(\theta) k - \pi_L (r_L + dt_L) - \pi_H (r_H + dt_H) + (1+\beta) da \\ + \beta \sum_{i=L,H} \pi_i (\mathbb{E}(\theta) k_i - \pi_L (r_{iL} + dt_{iL}) - \pi_H (r_{iH} + dt_{iH}) - dt_i),\end{aligned}$$

subject to:

1. Feasibility constraints at $t = 1$ and $t = 2$.

2. Incentive compatibility constraints at $t = 1$:

$$\begin{aligned} & -r_L - dt_L + \beta (\mathbb{E}(\theta) k_L - dt_L - \pi_L (r_{LL} + dt_{LL}) - \pi_H (r_{LH} + d_{tH})) \\ = & -r_H - dt_H + \beta (\mathbb{E}(\theta) k_H - dt_H - \pi_L (r_{HL} + dt_{HL}) - \pi_H (r_{HH} + dt_{HH})). \end{aligned}$$

3. Incentive compatibility constraints at $t = 2$[9]:

$$r_{iL} + dt_{iL} = r_{iH} + dt_{iH},\ i = L, H.$$

4. Participation constraint for the lender at $t = 1$:

$$\begin{aligned} kq + \beta (\pi_L k_L + \pi_H k_H) \leq & \ \pi_L (r_L + dt_L) + \pi_H (r_H + dt_H) \\ & + \beta \sum_{i=L,H} \pi_i (dt_i + \pi_L (r_{iL} + dt_{iL}) + \pi_H (r_{iH} + dt_{iH})). \end{aligned}$$

[9] Risk neutrality implies that the incentive compatibility constraints at $t = 1$ and $t = 2$ can be expressed as one constraint, as stated in points 2 and 3.

5. Participation constraint for the lender at $t = 2$:

$$k_i \leq \pi_L \left(r_{iL} + dt_{iL}\right) + \pi_H \left(r_{iH} + dt_{iH}\right), \, i = L, H.$$

In contrast to the case in which only short-term contracts are allowed, using long-term contracts allows the borrower to directly contract on the loan amounts at $t = 2$.

Proposition 11 *In the optimal long-term contract ϕ^* :*

1. *The loan amounts are:*

$$\begin{aligned} k^{**} &= \left(1 + \beta + \beta\pi_H \left(\mathbb{E}\left(\theta\right) - 1\right)\right) da, \\ k_H^{**} &= a, \quad \textit{and} \quad k_L^{**} = 0. \end{aligned}$$

2. *Repayments in terms of consumption good are:*

$$\begin{aligned} r_L^{**} &= 0, \quad r_H^{**} = r_L^{**} + da + \beta\left(\mathbb{E}\left(\theta\right) - 1\right) da, \\ r_{2L}^{**} &= 0, \quad \textit{and} \quad r_{2H}^{**} = \theta_L k_H^{**}. \end{aligned}$$

3. *Asset transfers are:*

$$t_L^{**} = a, \quad t_H^{**} = 0, \quad t_{2L}^{**} = 0, \quad \textit{and} \quad t_{2H}^{**} = a.$$

Proof. Note that the participation constraints for the lender will hold with equality. Using the incentive compatibility constraints and the participation constraint in the first period, the objective function for the borrower becomes:

$$\left(\mathbb{E}\left(\theta\right) - 1\right)\left(r_L + dt_L + \beta\left(dt_L + r_{2L} + dt_{2L}\right) + \beta\pi_H \mathbb{E}\left(\theta\right) d\left(t_L - t_H\right)\right) + \left(1 + \beta\right) da.$$

This implies:

$$\begin{aligned} t_L^{**} &= a, \quad t_H^{**} = 0, \quad t_{2L}^{**} = 0 \\ r_{2L}^{**} &= 0, \quad \text{and} \quad r_L^{**} = 0. \end{aligned}$$

Using these expressions in the participation and incentive compatibility constraints gives the contract in the proposition. ■

Corollary 5 *The optimal short-term contracts implement the optimal long-term contract.*

The proof follows directly from Proposition 11.

Remark. *The optimal contract described here resembles the optimal contract in Bolton and Scharfstein (1990). However, there are key differences between the assumptions and results in their paper and the ones presented here. In this paper the presence of a long-lived financial asset allows for lending to occur with short-term contracts. In fact, the existence of a long-lived financial assets implies that a sequence of short-term contracts can achieve the same allocation as the optimal long-term contract which, as in* Bolton and Scharfstein (1990), *involves the firm not being operated if the low state is realized. Moreover, in* Bolton and Scharfstein (1990), *the firm itself is pledged as collateral by allowing it to be liquidated in the event of a bad report. Therefore, the asset used as collateral is needed in order for the firm to operate and, thus, cannot be sold if it wants to continue operating. In my model, the project available to the borrower are inalienable and the financial asset that is used as collateral in equilibrium is unrelated to the operations of the firm. This implies that the borrower can choose to sell his assets and still be able to invest in his project, which is crucial to answer why assets are used as collateral instead of being sold to raise funds.*

6.11 Correlated dividends and returns

In the baseline model I assumed the financial asset was riskless. However, many risky assets are used as collateral in financial transactions. For example, mortgage back securities (MBS) were massively used as collateral to finance new mortgages or real estate related securities at the onset of the $2008-2009$ financial crisis. This extension allows for risky financial assets whose dividends may be correlated with the returns of the project operated by the borrower. I show that, consistent with the widely spread use of MBS as collateral, assets with payoffs highly correlated with the investment opportunity are better collateral.

Consider the baseline model presented in Section 2 with the only difference being that the asset's dividend is stochastic and it is potentially correlated with the return of the borrower's risky project. Although the dividends are paid at the end of each period, after the borrower invests in his risky project, they are known at the beginning of each period. This implies that there is only uncertainty about the dividend paid by the asset in the second period. The asset's dividend distribution is such that:

$$\mathbb{E}\left(d_2|\theta_1=\theta_i\right)=d_i, \text{ for } i=L,H,$$

and

$$\mathbb{E}\left(d_2\right)=\pi_H d_H+\pi_L d_L=\bar{d}.$$

Given a dividend d_2 at $t = 2$, the borrower's problem in the second period remains the same. The value of a borrower who holds a units of the asset at the beginning of the second period is:

$$V_2^B(a, d_2) = \mathbb{E}(\theta) d_2 a.$$

Analogously, the value for the lender of holding a units of the asset at the beginning of the second period is:

$$V_2^L(a, d_2) = d_2 a.$$

Given that the value functions for the borrower and the lender are linear in the dividend level, it is easy to see that all the results in the baseline model generalize to this extension. In particular, the asset's debt capacity becomes:

$$D = d_1 + \beta \bar{d} + \beta \pi_H \left(\frac{V_2^B(a, d_H)}{a} - \frac{V_2^L(a, d_H)}{a} \right).$$

The following proposition shows that assets with dividends that are more highly positively correlated with the borrower's project are better collateral.

Proposition 12 *Assets that have dividends that are more highly positively correlated with the risky project have a higher debt capacity, i.e.,*

$$\frac{\partial D}{\partial d_H|_{\bar{d}}} > 0.$$

Proof. Since $\frac{V_2^L(a,d)}{a} = d$,

$$\frac{\partial D}{\partial d_H|_{\bar{d}}} = \beta \pi_H (\mathbb{E}(\theta) - 1) > 0.$$

■

In the model, the financial asset partly resolves the non-contractibility of the return on the risky project by allowing the borrower to raise funds. Since in equilibrium the borrower values the asset more than the lender does, $V_2^B(a, d_H) - V_2^L(a, d_H)$ is the endogenous cost of defaulting on the promised amount, $r_H - r_L$, which allows the borrower to credibly commit to truthfully reporting the project's return. An asset that has dividends that are more highly positively correlated with the return of the risky project has a higher d_H, which implies a higher cost of default for borrowers. This higher cost of default allows the borrower to commit a larger amount of consumption goods in the high state and, thus, increases the debt capacity of the asset, which makes it better collateral.

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