Strategic Fragmented Markets*

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May 2017

Abstract

We study the determinants of asset market fragmentation. We develop a model of market formation with strategic investors that have heterogeneous valuations for an asset. Investors choose a dealer with whom to trade considering their price impact and the liquidity provided by the dealer. Fragmented markets are supported in equilibrium when investors' valuations are sufficiently correlated. In this case, liquidity provision is scarce and investors are more willing to accept a higher price impact. Dealers can benefit from fragmentation, but investors are always better off in centralized markets. Market fragmentation contributes to lower trading volumes relative to a centralized market.

JEL Codes: G12, D43, D47.

Keywords: market fragmentation, inter-dealer trading, demand schedule equilibrium.

*Email addresses: anababus@gmail.com, cparlato@stern.nyu.edu. We thank Darrell Duffie, Joel Hasbrouck, Ricardo Lagos, Guillermo Ordoñez, Thomas Sargent, Laura Veldkamp, Venky Venkateswaran and Pierre-Olivier Weill for their comments. The views expressed here need not reflect those of the Federal Reserve Bank of Chicago or the Federal Reserve System.

1 Introduction

Financial assets are traded in a wide variety of market structures. One of the main differences among financial markets is the degree of fragmentation they exhibit. For example—in increasing level of fragmentation—large stocks are traded in exchanges, Nasdaq operates as a dealer network, U.S. treasuries have been traditionally traded through primary dealers, and many debt instruments are traded over the counter (OTC). The structure of financial markets is crucial in determining their efficiency and their price dynamics. As such, the structure of financial markets has been the target of recent regulations and regulatory proposals. Yet, a fundamental question remains unsettled. What drives market fragmentation?

To answer this question, we develop a model of market formation with strategic investors that have heterogeneous valuations for an asset. In our environment, each investor can choose a dealer with whom to trade. Investors' choices determine a *market structure*. Investors are strategic both when they choose the market structure and when they trade. When choosing a dealer, an investor weighs her price impact against the dealers' willingness to provide liquidity. We show that fragmented markets are supported in equilibrium when investors' valuations for the asset are sufficiently correlated, even if there are no transaction costs. When valuations are highly correlated, dealers provide less liquidity. In this case, investors accept to trade in a market in which they have a higher price impact in exchange for a larger share of the dealer provided liquidity, and fragmentation arises in equilibrium.

The welfare implications of market fragmentation are distinct for dealers and investors. While dealers benefit from trading in a fragmented market provided investors' valuations are sufficiently dispersed, investors are always better off trading in a centralized market. Thus, trading in a fragmented market can be inefficient. Moreover, when fragmented market structures are supported in equilibrium, trading volume in a centralized market is higher than the one that would occur in fragmented markets, suggesting fragmentation itself decreases liquidity. Finally, conditional on market fragmentation arising in equilibrium, the degree of fragmentation is decreasing in investor valuations.

Our model has three dates and a finite number of strategic investors and dealers. Investors are ex-ante homogeneous but have ex-post heterogeneous valuations for an asset that is in zero-net supply. Dealers are homogeneous and do not value the asset intrinsically. Before investors' idiosyncratic asset valuations are realized, each investor chooses a dealer with whom to trade. Their choices determine a market structure. After the market structure is decided, trade takes place sequentially. First, each dealer and the investors that chose her trade in a local market. Second, dealers participate in an inter-dealer market. We model both investors' and dealers' trading strategies as quantity-price schedules. Moreover, when each agent chooses her trading strategy, she understands the impact of her trade on the price (taking all other agents' strategies as given). Thus, investors act strategically both when markets form, as well as when they trade.

There are three main assumptions in our model: a) the heterogeneity in asset valuations, b) the timing of trade, and c) the trading protocol. First, the heterogeneity in asset valuations can be interpreted as differences in liquidity needs, in the use of the asset as collateral, in risk management constraints or as disagreement about the value of the asset. Second, by assuming that trade takes place sequentially, we can study role of the inter-dealer market for intermediating trade and sharing risk. In fact, trade for various assets takes place in a similar set-up. For instance, Collin-Dufresne, Junge, and Trolle (2016) and Duffie, Scheicher, and Vuillemey (2015) find evidence that in the CDS market dealers use inter-dealer markets to manage inventory risk after trading with clients. Third, representing agents' strategies as quantity-price schedules allows us to capture common elements of the increasingly diverse set of trading protocols that are used in practice in decentralized markets. In particular, a robust feature of today's decentralized markets is that a relatively small number of dealers intermediate a vast proportion of transactions. This suggests that trading outcomes should reflect dealers' market power relative to other dealers as well as relative to investors.

We obtain three sets of results. The first set of results concentrates on understanding when fragmented market structures can arise in equilibrium. In our main theorem, we show that a fragmented market structure is an equilibrium when investors' valuations are sufficiently positively correlated. When markets are fragmented, dealers intermediate trade between the investors in their local market and the rest of the investors through the inter-dealer market. Even though dealers do not value holding the asset intrinsically, they are willing to trade with the investors in the local market and off-load their positions in the inter-dealer market to arbitrage any price differences between the two markets. However, a dealer's capacity to intermediate is limited by the price impact of her trades in her local market and in the inter-dealer market. Market fragmentation arises when investor valuations are sufficiently correlated, as competition for the dealer's liquidity in the local market intensifies. This effect dominates any price improvements that arise when more investors trade in the same local market. When investor valuations are dispersed, competition for the dealer's liquidity is low and having a lower a price impact dominates. Moreover, when we consider entry in the dealer market, conditional on markets being fragmented, the degree of fragmentation is decreasing in the correlation in investor valuations.

The second set of results focuses on the welfare properties of fragmented and centralized market structures. To model a centralized market, we use a standard approach and consider that all investors and dealers trade in the same market. We analyze investor and dealer welfare when they trade in a fragmented market and compare it to the welfare they would attain if they were to trade in a centralized market. We show that although dealers benefit from trading in a fragmented market provided investors' valuations are sufficiently dispersed, investors are always better off trading in a centralized market. Also, when investors' valuations are highly correlated, trading in a fragmented market is inefficient even if fragmentation is supported in equilibrium.

The third set of results concerns liquidity in fragmented and centralized markets. Our results confirm the intuition that assets that are traded in fragmented markets have intrinsically low liquidity, as proxied a by a high correlation between investors' valuations. However, the market structure itself further contributes to lowering the traded volume. Indeed, consistent with empirical findings, trading volume is lower in fragmented markets than in centralizes ones for the same level of dispersion in investors' valuations.

Literature Review

This paper relates to several strands of literature. The more relevant studies are those on endogenous market structure and intermediation in decentralized markets.

A series of papers have developed models in which the market structure in which an asset is traded is endogenously determined when traders incur transaction costs. An early contribution is Pagano (1989) who studies a set-up in which traders can choose to enter one of the two exchanges in which the same asset is traded. In Rust and Hall (2003) buyers and sellers choose to trade with a market maker, who posts publicly observable bid and ask prices, or with middlemen, who quote prices that are private information. In both models, traders concentrate in one market in the absence or fees or transaction costs.[1] In contrast, in our model markets fragmentation arises in equilibrium even when there are no exogenous trading costs.

A related set of papers narrows the focus and studies competition between venues. In Pagnotta and Philippon (2015) when two venues compete in the speed with which traders can find counterparties, markets fragmentation arises. Competition also plays a role in segmenting markets in Lester, Rocheteau, and Weill (2015), as dealers compete to attract order flow by posting the terms at which they execute trades.[2] We abstract from competition between venues. Instead, in our model investors choose a dealer with whom to trade based on the size of her local market, which is, in turn, determined by the other investors' choice.

Information asymmetries have been known to influence the market structure. For

[1] In Rust and Hall (2003) consumers and producers are indifferent between trading against a single middleman (concentration), or against a middleman and the market maker (fragmentation) at the Walrasian price.

[2] Although not directly concerned with studying market fragmentation, some other papers analyze models of competition between venues. These studies include Biais (1993), Glosten (1994), Hendershott and Mendelson (2000), Parlour and Seppi (2003), and Santos and Scheinkman (2001).

instance, in Zhu (2014) exchanges attract informed traders who want a fast execution of their order, while uninformed traders, who only have idiosyncratic liquidity needs, trade in dark pools. In contrast, in Kawakami (2016) trading in multiple venues is optimal to avoid excessive information aggregation as revealed risk cannot be traded away. In our model, there are no information asymmetries, and market fragmentation is driven by the dispersion of the investors' valuations for the asset.

Some recent papers address the question of which market structures facilitate efficient trade. In Malamud and Rostek (2014) agents may benefit from trading in interconnected venues relative to a centralized market. Duffie and Wang (2016) show that OTC markets can be efficient if agents write contingent bilateral contracts. Glode and Opp (2017) illustrate that a market in which agents face costly trading delays can be more efficient than a centralized market. In these models, the market structure is taken as given. In contrast, we focus on endogenizing the market structure.

There is a growing literature that studies the role of intermediaries in decentralized markets. Hugonnier, Lester, and Weill (2015), Neklyudov (2014) and Chang and Zhang (2016) propose models in which intermediaries facilitate trade between counterparties that otherwise would need to wait a long time to trade. Other papers explore the informational role of intermediaries. In Glode and Opp (2016) the role of intermediaries is to restore efficient trading by reducing adverse selection, while in Boyarchenko, Lucca, and Veldkamp (2016) inter-dealer information sharing improves risk sharing and welfare. Our model complements these works and shows that intermediaries' strategic trading behavior is a key determinant of market fragmentation.

We present the model in Section 2. In Section 3, we define and characterize the equilibrium of the model when markets are fragmented and, in Section 4, when the market is centralized. In Section 5 we compare welfare in fragmented and centralized markets while in Section 6 we focus on liquidity. We discuss other implications of our model in Section 8. Finally, we conclude in Section 9. All omitted proofs are in the Appendix.

2 The model

There are three dates, $t = 0, 1, 2$, and a finite number of agents that trade a risky asset in zero net supply. There are two types of agents, dealers and investors. There are $n_D \geq 3$ dealers indexed by $\ell = 1, ..., n_D$. We denote by N_D the set of dealers. The utility of a dealer who holds x units of the asset at time date 2 is given by

$$U_D(x) = -\frac{\gamma}{2}x^2.$$

There are also $n_I = n_S \cdot n_D$ investors indexed by $i = 1, ..., n_I$, where n_S is an integer and $n_S \geq 3$. The set of investors is denoted by N_I. An investor i derives utility

$$U_I^i(x) = \theta^i x - \frac{\gamma}{2}x^2$$

from holding x units of the asset at time 2, where

$$\theta^i = \theta + \eta^i,$$

with $\theta \sim N(0, \sigma_\theta^2)$, and $\eta^i \overset{i.i.d.}{\sim} N(0, \sigma_\eta^2)$. We refer to θ^i as investor i's valuation for the asset. The quadratic term in the utility functions for dealers and investors can be interpreted as the cost of holding the asset. The parameter γ plays the same role as a risk aversion coefficient.

Agents' preferences imply that dealers are homogeneous, while investors are heterogeneous in their valuation for the asset. The correlation among investors' valuations for the asset,

$$\rho \equiv Corr\left(\theta^i, \theta^j\right) \quad \forall i, j \in N_I, i \neq j,$$

captures the degree of heterogeneity across investors. It is immediate that $\rho = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\eta^2}$ with $\rho \in [0, 1]$. We consider that the degree of heterogeneity across investors is driven only by the idiosyncratic component of investors' valuations, η^i. In particular, we assume that changes in ρ reflect changes σ_η^2, while σ_θ^2 remains constant. As the dispersion of the idiosyncratic component of the investors' valuations, σ_η^2, goes to ∞, the heterogeneity among investors is maximal and $\rho \to 0$. In this case, investors have independent valuations. If

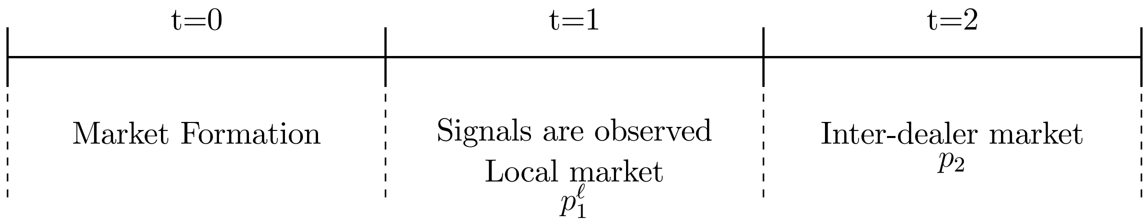

Figure 1: **Timing**

$\sigma_\eta^2 = 0$, the heterogeneity among investors vanishes and $\rho = 1$. This is the case when investors have a common valuation. The heterogeneity in investors' valuations of the asset captures, for instance, differences in liquidity needs, in the use of the asset as collateral, in risk management constraints, or disagreement about the value of the asset.[3]

The uncertainty about investors' valuations is realized at the beginning of date 1. The common component θ is observed by dealers and investors, whereas the idiosyncratic component η^i is the private information of investor i. We adopt this information structure to abstract from considerations related to learning from prices and focus on the agents' strategic trading behavior as a driver of market fragmentation.

Figure 1 illustrates the timing of the model. At date 0, each investor chooses a dealer with whom to trade. An investor can choose at most one dealer. However, multiple investors can choose the same dealer.[4] Once investors make their dealer selection, markets open and trade takes place. There are two rounds of trading. At date 1, each dealer ℓ trades strategically with the investors that have selected her at date 0, in a *local market* ℓ. At date 2, dealers trade strategically with each other in an *inter-dealer market*.

The investors' choices at date 0 determine a *market structure*, m, in which each dealer ℓ interacts with $n^\ell \geq 0$ investors. When an investor i chooses a dealer ℓ, we say that

[3]While these factors could play a role in dealers' valuation for an asset, we consider that heterogeneity across dealers is relatively less marked than across investors.

[4]Trade in many markets relies on relationships that are very concentrated. Hendershott et al. (2016) document that an investor in the corporate bond market trades on average with six dealers over the course of more than a decade.

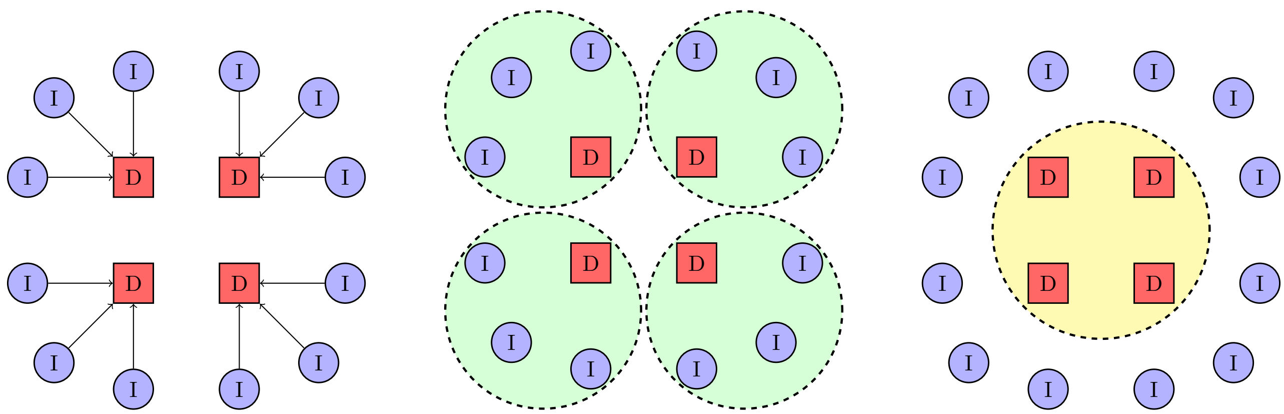

Figure 2: **Fragmented market structure**

$i\ell \in m$. We denote by $N_I(\ell)$ the set of investors that choose dealer ℓ. A fragmented market is *symmetric* when each dealer interacts with the same number of investors n_S. We denote by m_{n_S} a symmetric fragmented market. Figure 2 illustrates a fragmented market structure.

We model agents' trading strategies as quantity-price schedules. More precisely, the strategy of an agent is a map from her information set to the space of demand functions, as follows. The demand function of an investor i with an idiosyncratic valuation θ^i is a continuous function $X_1^i : \mathbb{R} \to \mathbb{R}$ which maps the price p_1^ℓ that prevails in the local market ℓ, into a quantity x_1^i she wishes to trade

$$X_1^i\left(p_1^\ell; \theta^i\right) = x_1^i. \tag{1}$$

A dealer's strategy has two components, each corresponding to the markets in which she trades. When trading in a local market ℓ at date 1, the demand function of a dealer ℓ who observes θ, is a continuous function $Q_1^\ell : \mathbb{R} \to \mathbb{R}$ which maps the price p_1^ℓ, into a quantity q_1^ℓ she wishes to trade

$$Q_1^\ell\left(p_1^\ell; \theta\right) = q_1^\ell. \tag{2}$$

At date 2, the demand function of a dealer ℓ who observed θ in the local market, is a continuous function $Q_2^\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which maps the quantity, q_1^ℓ, she acquired in the local market and the price that prevails in the inter-dealer market, p_2, into a quantity q_2^ℓ she wishes to trade

$$Q_2^\ell \left(p_2; \theta, q_1^\ell\right) = q_2^\ell. \tag{3}$$

Given a market structure m, the expected payoff for an investor i at date 0, corresponding to the strategy profile $\left\{X_1^i, \left(Q_1^\ell, Q_2^\ell\right)\right\}_{i \in N_I, \ell \in N_D}$ is

$$V_1^i (m) = \mathbb{E}_0 \left[\theta^i X_1^i \left(p_1^\ell; \theta^i\right) - \frac{\gamma}{2} \left(X_1^i \left(p_1^\ell; \theta^i\right)\right)^2 - X_1^i \left(p_1^\ell; \theta^i\right) p_1^\ell\right], \tag{4}$$

where each p_1^ℓ is the price at which local market ℓ clears, i.e., p_1^ℓ is such that

$$Q_1^\ell \left(p_1^\ell; \theta\right) + \sum_{i \in N_I(\ell)} X_1^i \left(p_1^\ell; \theta^i\right) = 0, \; \forall \; \ell \in N_D. \tag{5}$$

Similarly, the expected payoff of a dealer ℓ at date 0 is

$$V_1^\ell (m) = \mathbb{E}_0 \left[-\frac{\gamma}{2} \left(Q_1^\ell \left(p_1^\ell; \theta\right) + Q_2^\ell \left(p_2; \theta, q_1^\ell\right)\right)^2 - Q_2^\ell \left(p_2; \theta, q_1^\ell\right) \cdot p_2 - Q_1^\ell \left(p_1^\ell; \theta\right) \cdot p_1^\ell\right], \tag{6}$$

where p_2 is the price at which the inter-dealer market clears

$$\sum_{\ell \in N_D} Q_2^\ell \left(p_2; \theta, q_1^\ell\right) = 0. \tag{7}$$

3 Equilibrium

In this section, we define and characterize the equilibrium. We start by computing the equilibrium in the inter-dealer market at date 2 given a market structure m and the dealers' choices in the local market at date 1. Then, we characterize the equilibrium in the local market given a market structure m. Lastly, we look at the equilibrium conditions in the market formation game which determine the equilibrium market structure, m. All omitted proofs are in the Appendix.

Definition 1 *An equilibrium is a market structure* m *and demand functions* $\left\{Q_1^\ell, Q_2^\ell\right\}_{\ell \in N_D}$ *for dealers and* $\{X_1^i\}_{i \in N_I}$ *for investors such that given the pricing functions in Eq.*(5) *and Eq.*(7)

1. Q_2^ℓ *solves each dealer* ℓ*'s problem in the inter-dealer market at date* 2

$$\max_{Q_2^\ell} -\frac{\gamma}{2}\left(q_1^\ell + Q_2^\ell\left(p_2; \theta, q_1^\ell\right)\right)^2 - p_2 Q_2^\ell\left(p_2; \theta, q_1^\ell\right), \tag{8}$$

   given the other dealers' demand functions $\left\{Q_2^l\right\}_{l \in N_D, l \neq \ell}$*;*

2. Q_1^ℓ *solves each dealer* ℓ*'s problem in her local market at date* 1

$$\max_{Q_1^\ell} \mathbb{E}_1\left[-\frac{\gamma}{2}\left(Q_2^\ell\left(p_2; \theta, q_1^\ell\right) + Q_1^\ell\left(p_1^\ell; \theta\right)\right)^2 - p_2 Q_2^\ell\left(p_2; \theta, q_1^\ell\right)|\theta, p_1^\ell\right] - p_1^\ell Q_1^\ell\left(p_1^\ell; \theta\right), \tag{9}$$

   given investors' demand function in local market ℓ, $\{X_1^i\}_{i \in N_I(\ell)}$*;*

3. X_1^i *solves each investor* i*'s problem in the local market at date* 1

$$\max_{X_1^i} \theta^i X_1^i\left(p_1^\ell; \theta^i\right) - \frac{\gamma}{2}\left(X_1^i\left(p_1^\ell; \theta^i\right)\right)^2 - X_1^i\left(p_1^\ell; \theta^i\right) p_1^\ell, \tag{10}$$

   given dealer ℓ*'s and the other investors' demand function in local market* ℓ, Q_1^ℓ *and* $\left\{X_1^j\right\}_{j \in N_I(\ell), j \neq i}$*, respectively, and*

4. *no investor* i *benefits from deviating and joining a different local market, i.e., the expected payoff an investor receives from deviating to the market structure* $(m - i\ell + i\ell')$ *is not larger than the expected payoff she obtains in the market structure* m

$$V_1^i(m - i\ell + i\ell') \leq V_1^i(m) \text{ for all } i \in N_I \text{ and all } \ell \in N_D. \tag{11}$$

Since there is a finite number of agents, all agents trade strategically and take into account their price impact when submitting their demand. For the same reason, as condition 4 in Definition 1 shows, when an investor evaluates the benefit of leaving the local market ℓ and joining the local market ℓ', she understands that trading outcomes in the market structure $m - i\ell + i\ell'$ are different than in the market structure m. To keep our analysis tractable, we restrict our attention to equilibria in which the market structure is symmetric and agents have linear trading strategies.

3.1 Inter-dealer market

At date 2, after each dealer trades with her investors, the strategic inter-dealer market opens. A dealer ℓ enters the inter-dealer market with an inventory q_1^ℓ of the asset, which she acquired in the local market. Each dealer is privately informed about her inventory and knows that the other dealers trade optimally in their local markets. In the inter-dealer market, dealers choose their strategies taking as given the other dealers' demand functions, as well as the distribution of their inventories acquired at date 1.

We simplify the optimization Problem (8), which is defined over a function space, to finding the functions $Q_2^\ell\left(p_2;\theta,q_1^\ell\right)$ pointwise. To do this, we fix the realization of the set of idiosyncratic values $\{\eta^i\}_{i\in N_I}$. This maps into a realization of inventories $\left\{q_1^\ell\right\}_{\ell\in N_D}$ that dealers bring in the inter-dealer market. Then, we solve for the optimal quantity that dealer ℓ demands in the inter-dealer market as she takes as given the demand functions of the other dealers. This procedure allows us to derive the optimal demand function of dealer ℓ point by point. We formalize the argument below.

The first order condition for dealer ℓ is

$$\gamma\left(q_1^\ell+q_2^\ell\right)+p_2+\frac{\partial p_{2,-\ell}}{\partial q_2^\ell}q_2^\ell=0 \tag{12}$$

where $p_{2,-\ell}$ is the inverse residual demand of dealer ℓ implied by

$$\sum_{l\in N_D,l\neq\ell}Q_2^l\left(p_2;\theta,q_1^l\right)+q_2^\ell=0.$$

Since holding the asset has no intrinsic value for dealers, each dealer ℓ chooses q_2^ℓ to minimize her cost of holding the asset, net of any cash transfers in the inter-dealer market. The first two terms on the left-hand side of Eq.(12) represent the direct costs of demanding an additional unit of the asset in the inter-dealer market. The first term represents the marginal increase in the holding cost whereas the second term is the cost of purchasing an additional unit of the asset. Since the inter-dealer market is strategic, there is an additional, indirect cost of increasing the quantity demanded: the impact this quantity has on the market clearing price. The third term in the first order condition for dealers captures this indirect cost. The following proposition establishes the existence and uniqueness of the equilibrium in the inter-dealer market.

Proposition 1 ***(Existence and Uniqueness)*** *Given a market structure m and inventories $\{q_1^\ell\}_{\ell \in N_D}$, there exists a unique symmetric equilibrium in linear strategies in the inter-dealer market.*

The equilibrium in the inter-dealer market is straightforward. The first order condition in Eq.(12) implies that the demand function of a dealer ℓ is

$$Q_2^\ell \left(p_2; \theta, q_1^\ell\right) = -\frac{1}{\gamma + \lambda_2^\ell}\left(\gamma q_1^\ell + p_2\right), \tag{13}$$

where $\lambda_2^\ell \equiv \frac{\partial p_{2,-\ell}}{\partial q_2^\ell}$ is dealer ℓ's price impact in the inter-dealer market. In equilibrium, dealer ℓ's price impact is given by

$$\lambda_2^\ell = \frac{\gamma}{n_D - 2}.$$

The equilibrium price in the inter-dealer market is

$$p_2^* = -\gamma \frac{\sum_{l \in N_D} q_1^l}{n_D}. \tag{14}$$

From Eq.(13) and Eq.(14) it follows that the equilibrium quantity traded by dealer ℓ when the dealers' inventories are $\{q_1^l\}_{l \in N_D}$ is

$$q_2^{\ell *} = \frac{\gamma}{\gamma + \lambda_2^\ell}\left(\frac{\sum_{l \in N_D} q_1^l}{n_D} - q_1^\ell\right). \tag{15}$$

A dealer trades in the inter-dealer market for risk-sharing reasons. If dealers were risk neutral or if there was no heterogeneity in the dealers' inventories, there would be no trade in the inter-dealer market. The term between parenthesis in Eq.(15) represents the gains from trading in the inter-dealer market for an individual dealer. The larger the difference between the average inventory in the market and an individual dealer's inventory, the larger the amount the individual dealer will trade. If all dealers hold the same amount of inventory at the beginning of date 2, trade in the inter-dealer market breaks down.

However, even when there is scope for risk-sharing, a dealer restricts her trade because of its impact on the price. As the first term in Eq.(15) shows, a larger price impact λ_2^ℓ makes dealers less willing to trade. The price impact captures how strategic the inter-dealer market is and depends on the number of dealers participating in the market. The larger the number of dealers in the inter-dealer market, the less strategic the inter-dealer market is, and the lower the price impact of each dealer. Therefore, for a given level of heterogeneity in inventories, dealers will trade more in less strategic markets. As $n_D \to \infty$, and the market becomes perfectly competitive, all dealers hold the same amount of the asset at the end of period 2 irrespective of their inventory choice at date 1. In this case, dealers can share the idiosyncratic risk they face in the local markets perfectly.

3.2 Local markets

At date 1, after each investor chooses a dealer with whom to trade and all idiosyncratic valuations are realized, strategic local (investor-dealer) markets open. Each market ℓ is comprised of dealer ℓ and the n^ℓ investors who chose to trade with her. Each of these market participants chooses her trading strategy optimally taking the other participants' trading strategies as given. As in the inter-dealer market, we solve for the demand functions that solve the optimization Problem (10) for investors and Problem (9) for dealers pointwise.

Investors. The first order condition for an investor i in local market ℓ is

$$\theta^i - p_1^\ell - \gamma x^i - \frac{\partial p_{1,-i}^\ell}{\partial x_1^i} x_1^i = 0, \tag{16}$$

where $p_{1,-i}^\ell$ is the inverse residual demand function for investor i implied by

$$\sum_{j \in N_I(\ell), j \neq i} X_1^j \left(p_1^\ell; \theta^i\right) + x_1^i + Q_1^\ell \left(p_1^\ell; \theta\right) = 0.$$

Each investor i demands a quantity x_1^i so that her marginal utility equalizes her marginal cost of trading. The first term in Eq.(16) is the marginal benefit of increasing the final asset holdings for investor i, which is given by her valuation θ^i. The following three terms in Eq.(16) represent investor i's marginal cost of increasing her demand. The second and third terms represent the price the investor pays to acquire the asset and the increase in her holding costs, respectively. The last term is investor i's price impact, which captures the cost of trading in a strategic market.

Dealer. The first order condition for dealer ℓ in the local market is

$$\frac{dV_2^\ell(m)}{dq_1^\ell} - p_1^\ell - \frac{\partial p_{1,-\ell}^\ell}{\partial q_1^\ell} q_1^\ell = 0, \tag{17}$$

where $V_2^\ell(m)$ represents the payoff that dealer ℓ expects, at date 1, to receive in the inter-dealer market

$$V_2^\ell(m) = \mathbb{E}_1\left[-\frac{\gamma}{2}\left(q_2^{\ell*} + q_1^\ell\right)^2 - p_2 q_2^{\ell*} | \theta, p_1^\ell\right].$$

Since all agents are strategic, each dealer ℓ takes into account the effect of her trade on the inverse residual demand function, $p_{1,-\ell}^\ell$, in her local market which is implied by

$$\sum_{j \in N_I(\ell)} X_1^j\left(p_1^\ell; \theta, \eta^i\right) + q_1^\ell = 0.$$

Analogous to the investor's problem, dealer ℓ will demand a quantity q_1^ℓ to equalize the marginal benefit and the marginal cost associated with increasing the quantity demanded. Dealers do not attach any intrinsic value to the asset. However, since they can access both the local and inter-dealer markets, they can benefit from arbitraging price differences across both markets. The first term in Eq.(17) represents the expected marginal benefit of increasing the quantity that the dealer brings to the inter-dealer market. The second term captures the pecuniary cost of increasing q_1^ℓ and the third term captures the cost of trading in strategic markets, as reflected in the dealer's price impact. The next proposition establishes the existence and uniqueness of the equilibrium in the local markets.

Proposition 2 ***(Existence and uniqueness)*** *Given a market structure m, there exists a unique symmetric equilibrium in linear strategies at date* 1.

To compute the equilibrium at date 1, we derive the equilibrium trading strategies of the investors and the dealer in each local market ℓ. The first-order condition in Eq.(16)

implies that the demand function of an investor i in local market ℓ is

$$X_1^i\left(p_1^\ell;\theta^i\right) = \frac{1}{\gamma+\upsilon_1^\ell}\left(\theta^i - p_1^\ell\right), \tag{18}$$

where $\upsilon_1^\ell \equiv \frac{\partial p_{1,-i}^\ell}{\partial x_1^i}$ is investor i's price impact in local market ℓ. Similarly, the first-order condition for dealer ℓ in her local market in Eq.(17) implies that the demand function of the dealer ℓ is

$$Q_1^\ell\left(p_1^\ell;\theta\right) = \frac{1}{\lambda_1^\ell}\left(\frac{dV_2^\ell(m)}{dq_1^\ell} - p_1^\ell\right), \tag{19}$$

where $\lambda_1^\ell \equiv \frac{\partial p_{1,-\ell}}{\partial q_1^\ell}$ represents dealer ℓ's price impact in her local market. The quantity that an agent trades in a local market is proportional to her perceived marginal gain of holding the asset, which is given by the difference between her marginal valuation for the asset and the price of the asset in the local market. An investor's marginal valuation is simply θ^i, while a dealer's marginal valuation for the asset is given by her expected payoff from trading an additional unit of the asset in the inter-dealer market, $\frac{dV_2^\ell(m)}{dq_1^\ell}$. As in the inter-dealer market, the price impact in the local market restricts the amount traded by dealers and investors. The following lemma characterizes the investors' and dealer's equilibrium price impact in the local market.

Lemma 1 ***(Equilibrium price impacts)*** *In each local market ℓ, the investors' and the dealer's equilibrium price impact satisfy the following system of equations*

$$\upsilon_1^\ell = \frac{1}{\frac{n^\ell-1}{\gamma+\upsilon_1^\ell} + \frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}}, \tag{20}$$

$$\lambda_1^\ell = \frac{1}{\frac{n^\ell}{\gamma+\upsilon_1^\ell}}. \tag{21}$$

Substituting the investors' and dealer's demand functions, respectively in Eq.(18) and Eq.(19), into the market clearing condition in Eq.(5), and using the expression for the dealer's price impact in Eq.(21), the price in the local market is

$$p_1^\ell = \frac{1}{2}\frac{dV_2^\ell}{dq_1^\ell} + \frac{1}{2}\frac{\sum_{i\in N_I(\ell)}\theta^i}{n^\ell}. \tag{22}$$

Eq.(22) shows that the equilibrium price in the local market is an average of the investors' average valuation, $\frac{\sum_{i \in N_I(\ell)} \theta^i}{n^\ell}$, and the dealer's expected marginal valuation, $\frac{dV_2^\ell}{dq_1^\ell}$.

As can be seen from Eq.(20) and Eq.(21) in Lemma 1, the price impact for the investors, υ_1^ℓ, and for the dealer, λ_1^ℓ, in local market ℓ depend on the market structure m, only through the number of dealers, n_D, participating in the inter-dealer market. Lemma B.9 and Lemma B.10 in the Appendix characterize the price impact for dealers and investors in the local markets. These lemmas show that investors and dealers trade more aggressively when local markets are larger, and the inter-dealer market is less strategic. The larger the number of investors in the local market, the lower the price impact and the more investors and dealers react to the price. When the inter-dealer market is less strategic, it is less costly for dealers to unload their inventory at date 2 and they are willing to provide more liquidity in their local market. This translates into a higher asset supply in the local market, which leads to investors trading more aggressively.

The market structure affects the quantity that the dealer and her investors trade in the local market through the dealer's marginal valuation of the asset. Indeed, the dealer's marginal valuation of the asset depends on the quantity she expects to buy in the inter-dealer market, which depends on the price she expects to face at date 2. Differentiating V_2^ℓ, using Eq.(12) and substituting in dealer ℓ's demand function in the local market in Eq.(19), as well the price in the local market in Eq.(22), we obtain that the dealer's expected marginal valuation for the asset is given by

$$\frac{dV_2^\ell}{dq_1^\ell} = (1-w)\,\frac{\gamma}{\gamma+\lambda_2^\ell}\mathbb{E}\left[p_2|\theta, p_1^\ell\right] + w\frac{\sum_{i\in N_I(\ell)}\theta^i}{n^\ell}, \tag{23}$$

where

$$w = \frac{\gamma\lambda_2^\ell}{2\gamma\lambda_1^\ell + \gamma\lambda_2^\ell + 2\lambda_1^\ell\lambda_2^\ell}.$$

The expected price at date 2, in turn, depends on the market structure, which determines the extent of the risk-sharing opportunities among dealers in the inter-dealer market. More specifically, dealer ℓ expects the price in the inter-dealer market to be

$$\mathbb{E}\left(p_2|\theta, q_1^\ell\right) = -\gamma\frac{1}{n_D}\left(q_1^\ell + \sum_{l\in N_D, l\neq \ell}\mathbb{E}\left[q_1^l|\theta\right]\right). \tag{24}$$

The expression in Eq.(24) depends on all the other dealers' trades in their local markets, which, in turn, depend on the conditions in their local markets, i.e., the number of participants and the equilibrium trading strategies.

The degree of heterogeneity in investors' valuations for the asset, ρ, also affects the quantity that agents trade, and, as we show in Theorem **1**, the equilibrium market structure. To keep the level of aggregate uncertainty constant, we interpret differences in the correlation among investor valuations as being driven only by differences in the volatility in the idiosyncratic component of investors' valuations, and keep the volatility of the common component constant. Using the expression for the equilibrium price in local market ℓ in Eq.(22) in the investors' demand in Eq.(18), we find that the quantity that an investor i trades in the local market ℓ can be written as

$$x_1^i = \frac{1}{\gamma + \upsilon_1^\ell}\left[\frac{1}{2}\left(\theta^i - \frac{dV_2^\ell}{dq_1^\ell}\right) + \frac{1}{2}\left(\theta^i - \frac{\sum_{i\in N_I(\ell)}\theta^i}{n^\ell}\right)\right]. \tag{25}$$

The term $\left(\theta^i - \frac{dV_2^\ell}{dq_1^\ell}\right)$ represents the gains that the investor obtains from trading with the dealer, while the term $\left(\theta^i - \frac{\sum_{i\in N_I(\ell)}\theta^i}{n^\ell}\right)$represents the gains that the investor obtains from trading with other investors. Thus, as investors valuations become more correlated and $\rho \to 1$, the gains from trading with other investors decrease. At the same time, when investors' valuations are highly correlated so are the dealers' positions entering the inter-dealer market, which decreases the risk-sharing opportunities between them and the dealers' willingness to provide liquidity in their local markets. Thus, as $\rho \to 1$ the gains from trading with the dealer decrease as well.

Moreover, when more investors participate in the local market the price impact of the dealer, λ_1^ℓ, decreases. Thus, trading in the local market is more beneficial for the dealer. This can be seen by inspecting the dealer's expected marginal valuation in Eq.(**23**), which shows that the average investor valuation becomes relatively more important for the dealer than the price she expects to trade at in the inter-dealer market, when the number of investors in her local market increases. Indeed, as $n^\ell \to \infty$, $\lambda_1^\ell \to 0$ and $w \to 1$. This implies that when ρ approaches 1, an investor's gains from trading with the dealer vanish if $n^{\ell} \to \infty$.

3.3 Market Formation

At date 0, before all information is revealed, each investor i chooses a dealer with whom to trade. Since each investor i takes the other investors' choices as given, from investor i's perspective, choosing a dealer with whom to trade is the same as choosing between two market structures. A symmetric fragmented market structure is an equilibrium if

$$\Delta^{i}(\rho; n_D) \equiv V_1^{i}(m_{n_S}) - V_1^{i}(m_{n_S} - i\ell + i\ell') > 0, \forall i \in N_I \tag{26}$$

where $\Delta^{i}(\rho; n_D)$ represents the marginal benefit for investor i of participating in symmetric market structure, m_{n_S}, relative to participating in market structure $(m_{n_S} - i\ell + i\ell')$, when there are n_D dealers and when the correlation across investor valuations is ρ. Effectively, an investor weighs the benefit of trading in a local market ℓ against one dealer and other $(n_S - 1)$ investors, against the benefit of trading in a local market ℓ against one dealer and other n_S investors. In doing so, she anticipates that dealers trade in the inter-dealer market. In particular, since the equilibrium in the inter-dealer market depends on the market structure, changing the size of the local market impacts the dealers' behavior both directly, and indirectly, by changing the risk-sharing opportunities in the inter-dealer market. This feeds back into the investor's decision whether to deviate or not. The following theorem states when symmetric market structures arise in equilibrium.

Theorem 1 ***(Market fragmentation)*** *There exists a threshold $\rho^* \in [0, 1)$ and a number of dealers $n_D^*(\rho^*, n_I)$ such that for all $\rho > \rho^*$ the fragmented symmetric market structure with $n_D^*(\rho^*, n_I)$ dealers and n_I investors is an equilibrium.*

Theorem 1 shows that if investor valuations are sufficiently positively correlated a fragmented market structure is an equilibrium and no individual investor has incentives to deviate to a larger local market. A larger local market has two opposing effects on an investor's expected profit. On the one hand, having more investors in the local market reduces the investor's price impact and allows her to trade more. On the other hand, the larger the number of investors, the steeper the competition for the dealer's liquidity. When ρ approaches one, the gains from trade with other investors in the local market disappear. At the same time, the larger the correlation between investors' valuations the lower the dispersion in dealers' inventories and the lower the gains from trade for a dealer in the inter-dealer market. This decreases the dealers' willingness to supply liquidity in their local markets, for which investors compete, and reduces investor's gains from trade with the dealer as well. For sufficiently large values of ρ, the loss in gains from trade outweigh the benefits from having a smaller price impact and deviating from a fragmented symmetric market structure is not profitable for any investor.

While it is investors' choices that determine the market structure, dealers also play a crucial role. The following proposition illustrates the effect that dealers' trading has on the equilibrium market structure.

Proposition 3 ***(Competitive inter-dealer market)*** *No fragmented symmetric market structure can be supported in equilibrium as $n_D \to \infty$.*

Proposition 3 shows that dealers strategic trading behavior is a key determinant of market fragmentation. Indeed, the inter-dealer market becomes perfectly competitive as $n_D \to \infty$. In this case, the price impact of each dealer in the inter-dealer market goes to 0. Since it becomes cheaper for dealers to intermediate, they are willing to provide more liquidity in their local markets. Joining a larger local market becomes more attractive for an investor, as she benefits from a lower price impact without changing the competition for the dealer's liquidity. Formally, we have that $\lim_{n_D \to \infty} \Delta^i (\rho; n_D) < 0$.

4 A Benchmark: Centralized markets

A useful benchmark to study the implications of market fragmentation is a centralized market. We consider that in a centralized market structure, m_c, trade takes place between all investors and dealers only at date 1. A centralized market can arise in our set-up when no investor chooses a dealer with whom to trade at date 0.

Just as in the case of fragmented markets, agents trading strategies in a centralized market are represented by price-quantity schedules. In particular, the strategy of an agent is a map from her information set to the space of demand functions, as follows. The demand function of an investor i with an idiosyncratic valuation θ^i is a continuous function $X_c^i : \mathbb{R} \rightarrow \mathbb{R}$ which maps the price p_1^c that prevails in the centralized market, into a quantity x_c^i she wishes to trade. Similarly, the demand function of dealer ℓ who observes θ, is a continuous function $Q_c^\ell : \mathbb{R} \rightarrow \mathbb{R}$ which maps the price p^c, into a quantity q_c^ℓ she wishes to trade.

The expected payoff for an investor i at date 0, corresponding to the strategy profile $\left\{X_c^i, Q_c^\ell\right\}_{i \in N_I, \ell \in N_D}$ is

$$V_c^i\left(m_c\right) = \mathbb{E}_0\left[\theta^i X_c^i\left(p_c; \theta^i\right) - \frac{\gamma}{2}\left(X_c^i\left(p_c; \theta^i\right)\right)^2 - X_c^i\left(p_c; \theta^i\right) p_c\right],$$

while the expected payoff of a dealer ℓ at date 0 is

$$V_c^\ell\left(m_c\right) = \mathbb{E}_0\left[-\frac{\gamma}{2}\left(Q_c^\ell\left(p_c; \theta\right)\right)^2 - Q_c^\ell\left(p_c; \theta\right) p_c\right],$$

where each p_c^c is the price at which the market clears, or

$$\sum_{i \in N_I} X_c^i\left(p_c; \theta^i\right) + \sum_{\ell \in N_D} Q_c^\ell\left(p_c; \theta\right) = 0.$$

In a centralized market, all of the $n_I + n_D$ market participants solve the same problem as the one solved by an investor in a fragmented market structure. The first order condition for an investor $i \in N_I$ is

$$\left(\theta^i - p_c\right) - \left(\gamma + \frac{\partial p_{c,-i}}{\partial x_c^i}\right) x_c^i = 0,$$

while the first order condition for a dealer $\ell \in N_D$ is

$$\left(-p_c\right) - \left(\gamma + \frac{\partial p_{c,-\ell}}{\partial q_c^\ell}\right) q_c^\ell = 0$$

where $p_{c,-h}$ is the inverse residual demand of market participant h implied by

$$\sum_{\substack{j \in N_I \\ j \neq h}} X_c^j \left(\theta^j, p_c\right) + \sum_{\substack{\ell \in N_D \\ \ell \neq h}} Q_c^\ell \left(p_c; \theta\right) + y_c^h = 0,$$

where y_c^h is the quantity demanded by market participant h.

Proposition 4 ***(Existence and uniqueness)*** *Given a centralized market structure m_c there exists a unique symmetric equilibrium in linear strategies at date 1.*

As Proposition 4 shows, there is a unique equilibrium at date 1 in a centralized market structure. In this equilibrium, the demand function for an investor i is given by

$$X_c^i \left(\theta^i, p_c\right) = \frac{\theta^i - p_c}{\gamma + \lambda_c}, \tag{27}$$

while the demand function for a dealer ℓ is given by

$$Q_c^\ell \left(p_c; \theta\right) = \frac{-p_c}{\gamma + \lambda_c} \tag{28}$$

where $\lambda_c \equiv \frac{\partial p_{1,-h}^c}{\partial x_c^i}$ is the price impact of participant h in the centralized market. In equilibrium, this price impact is given by

$$\lambda_c = \frac{\gamma}{n_I + n_D - 2},$$

and the price is

$$p_c = \frac{\sum_{j \in N_I} \theta^j}{n_I + n_D}. \tag{29}$$

As in the inter-dealer market, the equilibrium price in Eq.(29) in the centralized market is equal to the average valuation of the market's participants. Putting together Eq.(27) and Eq.(29) we can see that a market participant h will hold a larger position the larger the difference between her private valuation θ^h and the average valuation in the market.

5 Welfare

In this section, we compare investor and dealer welfare in fragmented and centralized market structures. For all the numerical illustrations in this section, we consider the case with $n_D = 7$, $n_S = 10$, $\gamma = 1$ and $\sigma_\theta^2 = 1$.

5.1 Investor welfare

The expected welfare of an investor i who chooses a dealer ℓ given a symmetric market structure m_{n_S} when the correlation in investor valuations is ρ is given by

$$V_1^i\left(m_{n_S}\right) = \mathbb{E}_0\left[\theta^i x_1^i - \frac{\gamma}{2}\left(x_1^i\right)^2 - p_1^\ell x_1^i\right],$$

where x_1^i and p_1^ℓ are, respectively, the quantity of the asset the investor purchases and the price she pays for this quantity in equilibrium. The investor's expected welfare in a centralized market is

$$V^i\left(m_c\right) = \mathbb{E}_0\left[\theta^i x_c^i - \frac{\gamma}{2} x_c^i - p_c x_c^i\right],$$

where x_c^i and p_c are, respectively, the quantity of the asset the investor purchases and the price she pays for this quantity in equilibrium.

Lemma 2 ***(Risk sharing gains)*** *For a given symmetric market structure m_{n_S}, an investor's welfare is continuous and monotonically decreasing in ρ, i.e., $\frac{\partial V_1^i\left(m_{n_S};\rho\right)}{\partial\rho} < 0$.*

Lemma 2 shows that the higher the correlation in investor valuations, the lower the investors' expected welfare. A higher correlation in investor valuations implies less risk-sharing opportunities between investors and between dealers in the inter-dealer market. The lower these risk-sharing opportunities, the lower the expected gains from trade and the lower the investors' expected welfare.

Let $\Delta V^i = V^i\left(m_c\right) - V_1^i\left(m_{n_S}\right)$ be the difference between investors' expected utility in centralized and fragmented numbers with the same number of dealers. The following proposition compares investor welfare in centralized and fragmented markets.

Proposition 5 ***(Investor welfare)*** *Investors are always better off in a centralized market structure, i.e, $\Delta V^i > 0\ \forall\rho$.*

As Proposition 5 shows, investors are always better off in a centralized market structure than in a fragmented one. Figure 3 illustrates this result. The blue solid line represents investor welfare in fragmented markets, the red dashed line represents investor welfare

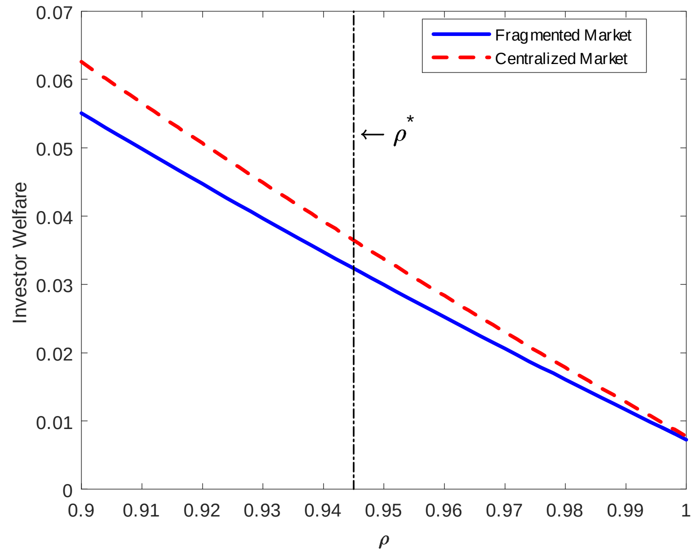

Figure 3: Investor welfare in centralized and fragmented markets as a function of ρ for $n_D = 7$, $n_S = 10$, $\gamma = 1$ and $\sigma_\theta^2 = 1$.

in centralized markets and the vertical dotted line is the threshold ρ^* above which the symmetric fragmented market structure is an equilibrium. A centralized market structure reduces the investors' price impact and it increases the expected gains from trade. When all investors trade in the same market, the expected gains from trade for an investor are larger than when her trade with investors in other local markets is intermediated by dealers. Though the effect on the expected gains from trade disappears when $\rho = 1$, the lower price impact associated with trading in a bigger market is always present and investors are strictly better off in a centralized market structure.

5.2 Dealer welfare

A dealer's expected welfare in a symmetric fragmented market structure is

$$V_1^\ell\left(m_{n_S}\right) = -\mathbb{E}\left[\frac{\gamma}{2}\left(q_1^\ell + q_2^\ell\right)^2 + p_1^\ell q_1^\ell + p_2 q_2^\ell\right],$$

where q_1^ℓ and q_2 are, respectively, the quantities the dealer buys in the local market at price p_1^ℓ and in the inter-dealer market at price p_2, in equilibrium.

In a centralized market structure, a dealer's expected welfare is

$$V_c^{\ell}(m_c) = -\mathbb{E}\left[p_c q_c + \frac{\gamma}{2}(q_c)^2\right],$$

where q_c is the quantity the dealer purchases at price p_c in the market, in equilibrium.

Lemma 3 ***(Gains from intermediation)*** *For a given symmetric market structure m_{n_S}, a dealer's welfare is continuous and monotonically decreasing in ρ, i.e., $\frac{\partial V_1^{\ell}(m_{n_S},\rho)}{\partial \rho} < 0$.*

Lemma 3 states that dealer welfare is decreasing in the correlation in investor valuations, ρ. A higher correlation in investor valuations leads to a smaller price dispersion among the local markets. The only difference among dealers in their local markets in a symmetric market structure is the local price they face. Therefore, the more similar the prices in the local markets, the more similar the inventories dealers carry to the inter-dealer market and the smaller the gains for dealers from intermediating trade between the local markets through the inter-dealer market.

Let $\Delta V^D = V_c^{\ell}(m_c) - V_1^{\ell}(m_{n_S})$ be the welfare gain for dealers from trading in a centralized market. The next proposition compares dealer welfare in centralized and fragmented markets.

Proposition 6 ***(Dealer welfare)*** *There exists a $\rho_W \in (0,1)$ such that for $\rho < \rho_W$ dealers are better off in fragmented markets and for $\rho > \rho_W$ dealers are better off in centralized markets, i.e.,*

$$\Delta V^D \begin{cases} > 0 & \text{if } \rho > \rho_W \\ = 0 & \text{if } \rho = \rho_W \\ < 0 & \text{if } \rho < \rho_W \end{cases}.$$

Proposition 6 shows that dealers benefit from trading in a fragmented market structure when investor valuations are very dispersed and in a centralized market when investor valuations are highly correlated. Trading in a fragmented market structure allows dealers to profit from intermediating trades between investors in different local markets through

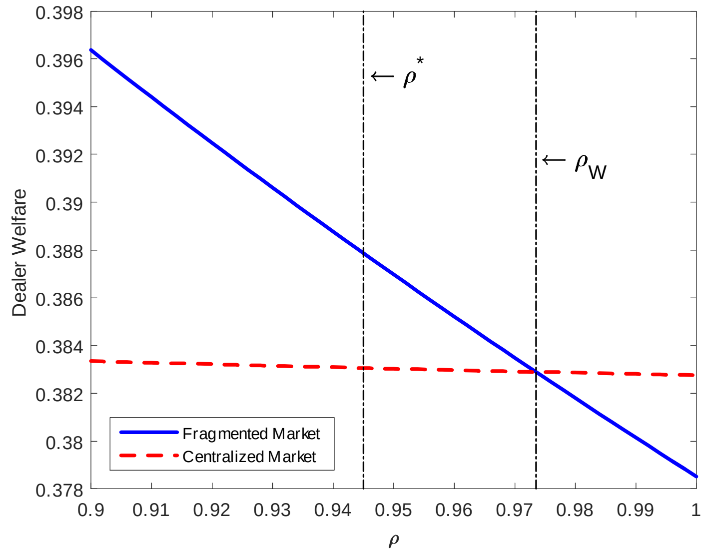

Figure 4: Dealer welfare in centralized and fragmented markets as a function of ρ for $n_D = 7$, $n_S = 10$, $\gamma = 1$ and $\sigma_\theta^2 = 1$.

the inter dealer market. The profits from intermediation disappear when the market structure is centralized. However, a centralized market offers a lower price impact than a fragmented one, making it cheaper for dealers to achieve their desired positions. When investor valuations are very dispersed, $\rho < \rho_W$, the dealers' profits from intermediation are high and dominate the higher price impact associated with fragmented market structures. In this case, dealer are better off in fragmented market structures. When investor valuations are very similar to each other and $\rho > \rho_W$, the dealers' profits from intermediating are small and the effect on price impact dominates. In this case, dealers are strictly better off trading in a centralized market.

Figure 4 illustrates a dealer's expected welfare in a fragmented market structure and in a centralized market structure. The blue solid line represents dealer welfare in fragmented markets which, consistent with Lemma 3, is decreasing in ρ. The red dashed line represents dealer welfare in centralized markets and the vertical dotted lines are the threshold ρ^* above which the symmetric fragmented market structure is an equilibrium and the threshold ρ_W above which dealers are better in a centralized market, respectively.

5.3 Efficiency

From Propositions 5 and 6 it follows that investors' and dealers' benefit differently from different market structures. When the dispersion in investor valuations is not sufficiently high, so that $\rho < \rho_W$, dealers are better off in fragmented markets, while investors are better off in centralized ones. If investor valuations are not very dispersed, so that $\rho > \rho_W$, both investors and dealers are better off trading in a centralized market than trading in a fragmented one, and a fragmented symmetric market structure is inefficient.

If $\rho^* > \rho_W$, a fragmented market is inefficient even if it is supported in equilibrium. The inefficiency is due to a coordination failure which prevents investors from choosing to trade in a centralized market structure. Indeed, when $\rho > \rho^*$, there is no benefit for an individual investor from deviating from a fragmented market structure m_{n_S} given that all other investors trade in the structure m_{n_S}.

6 Liquidity

In this section, we study the implication of our model for liquidity. We start by looking at the liquidity of the inter-dealer market, which measures the amount of intermediation in the economy, and then focus on the liquidity of date 1 markets both in fragmented and centralized markets.

6.1 Intermediated volume

Dealers act as intermediaries between investors in different local markets through the inter-dealer market. The amount traded in the inter-dealer market is the intermediated volume in a fragmented market structure. The expected average trading volume in the inter-dealer market is

$$\mathcal{V}_D\left(m_{n_S}\right) = \frac{1}{2}\mathbb{E}\left[\frac{\sum_{\ell=1}^{n_D}\left|q_2^\ell\right|}{n_D}\right].$$

As argue above, the dealers' willingness to intermediate decreases as the correlation in investors' valuations approaches 1. Lemma 4 formalizes this intuition and shows that

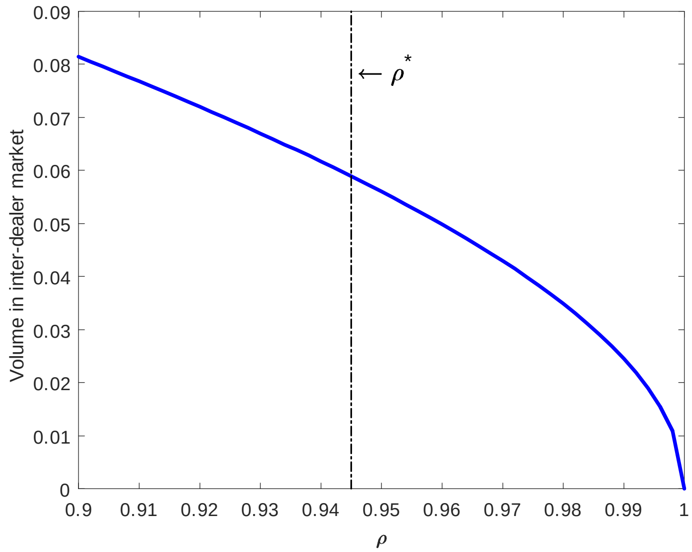

Figure 5: Intermediated volume in fragmented markets as a function of ρ for $n_D = 7$, $n_S = 10$, $\gamma = 1$ and $\sigma_\theta^2 = 1$.

intermediated volume decreases with the correlation in investor valuations and goes to zero as this correlation goes to 1.

Lemma 4 ***(Intermediated Volume)*** *In a symmetric fragmented market structure, the volume intermediated by dealers in the inter-dealer market decreases with the correlation in investor valuations ρ, $\frac{\partial \mathcal{V}_D}{\partial \rho} < 0$ with $\lim_{\rho \to 1} \mathcal{V}_D = 0$.*

When markets are fragmented, a higher correlation in investor valuations leads to a smaller price dispersion in local markets. This smaller dispersion implies a smaller dispersion in dealer inventories and, thus, decreases risk-sharing in the inter-dealer market, which leads to a lower inter-dealer volume, as shown by Lemma **4**. As the correlation in investor valuation approaches 1, price dispersion among local markets is 0 and intermediated volume goes to 0.[5] As we discuss in Section 5, when $\rho = 1$ and intermediated volume

[5]Note that the observation that intermediated volume goes to 0 as ρ goes to 1 holds only in a symmetric fragmented market. Typically, trade in the interdealer market is positive if there is a different number of investors in each local market, even when $\rho = 1$.

is zero, dealers do not profit from intermediation and they are better off in a centralized market than in a fragmented one. Figure 5 shows intermediated volume as a function of the correlation in investor valuations ρ.

6.2 Volume

The liquidity of markets at date 1 depends on the market structure. In a fragmented market structure, expected average trading volume in local market ℓ is given by

$$\mathcal{V}^{\ell}\left(m_{n_S}\right)=\frac{1}{2}\mathbb{E}\left[\frac{\sum_{i\in N(\ell)}\left|x_1^i\right|+\left|q_1^{\ell}\right|}{n^{\ell}+1}\right].$$

Analogously, in a centralized market structure, expected average trading volume is given by

$$\mathcal{V}_c=\frac{1}{2}\mathbb{E}\left[\frac{\sum_{i=1}^{n_I}\left|x_c^i\right|+n_D\sum_{\ell=1}^{n_D}\left|q_c^{\ell}\right|}{n_I+n_D}\right].$$

The amount of trade in local markets also depends on the correlation in investor valuations. Lemma 5 shows that volume decreases with the correlation in investor valuations in symmetric market structures, regardless of whether the market is fragmented.

Lemma 5 ***(Volume)*** *Expected trading volume in local markets decreases with the correlation in investor valuations ρ*

a) in the local markets in a fragmented symmetric market structure, $\frac{\partial\mathcal{V}^{\ell}}{\partial\rho}<0$, and

b) in a centralized market structure, $\frac{\partial\mathcal{V}^{c}}{\partial\rho}<0$.

As can be seen from Lemma 5, the more similar the valuations for the asset among investors, the lower the gains from trade among investors and, thus, the lower the incentives to trade in the market at date 1.

Figure 6 depicts date 1 trading volume in centralized and fragmented symmetric market structures. The blue solid line represents the expected average volume in a fragmented market at date 1 measured as the average of the expected volume in all local markets, $\sum_{\ell\in n_D}\frac{\mathcal{V}^{\ell}}{n_D}$. The red dashed line is the expected average volume in a centralized market

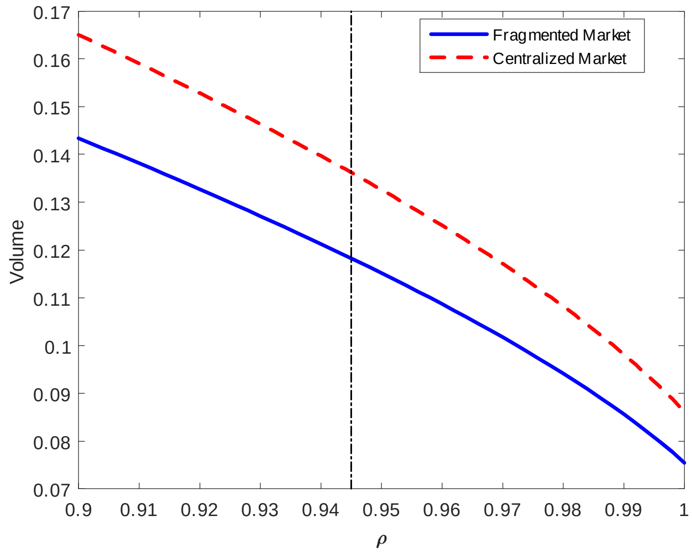

Figure 6: Volume at date 1 in fragmented and centralized markets as a function of ρ for $n_D = 7$, $n_S = 10$, $\gamma = 1$ and $\sigma_\theta^2 = 1$.

and the dotted black line is the threshold ρ^* above which a symmetric fragmented market is an equilibrium.

Lemma 5, together with Theorem 1, confirms the intuition that assets that are traded in fragmented markets have intrinsically low liquidity, proxied by high ρ. However, Figure 6 shows that the market structure itself also contributes to low volumes associated with decentralized markets. For a given correlation in investor asset valuations ρ, the volume traded is lower in fragmented markets than in centralized ones.

7 Dealer entry

A key determinant of the market structure in equilibrium is the degree of fragmentation in the economy. The degree of fragmentation is determined by the number of dealers n_D that provide intermediation services. So far, we have taken the degree of market fragmentation as given. However, different assets are traded in markets with different degrees of fragmentation. In this section, we extend the model in Section 2 to allow for dealer entry.

This extension yields interesting predictions that tie the degree of fragmentation to the primitives of the economy.

More specifically, we consider a large number of dealers $\bar{N}$ that can enter the dealer market at a fixed cost F. After dealers make their entry decision, the game is identical to the one described in Section 2. At $t = -1$ dealers make their entry choice, at $t = 0$ investors choose a dealer with whom to trade, at $t = 1$ each dealer trades with the investors that chose her and at $t = 2$ all dealers that entered trade in the inter-dealer market. The market structure is determined by the dealers' decision at $t = -1$ and by the investors' decisions at $t = 0$. While investors' choices determine whether fragmentation is an equilibrium, as we discuss in Section 3, the dealers' choices to enter the market determine the degree of fragmentation on the market.

Since there is a finite number of investors and a finite number of dealers, a strictly symmetric market structure, one in which all dealers have the same number of investors, does not always exist. To address the non-divisibility of investors we will focus on market structures in which the dispersion in the distribution of investors across dealers is minimized. We refer to these market structures as generalized symmetric.

Definition 2 *A generalized symmetric market structure with n_I investors and n_D dealers is a market structure in which x dealers have $\left\lfloor \frac{n_I}{n_D} \right\rfloor$ investors and $n_D - x$ have $\left\lfloor \frac{n_I}{n_D} \right\rfloor + 1$, where*

$$x = n_D \left(\left\lfloor \frac{n_I}{n_D} \right\rfloor + 1 \right) - n_I.$$

A dealer ℓ who has n^ℓ investors in a generalized symmetric market structure with n_I investors and n_D dealers has expected utility

$$V^\ell \left(n^\ell; n_I, n_D \right),$$

where $V^\ell \left(n^\ell; n_I, n_D \right) = V_1^\ell \left(m_G \right)$ as defined in Eq. 6 and m_G is the generalized symmetric market structure with n_I investors and n_D dealers. We assume that all dealers face the same probability of getting probability of having $\left\lfloor \frac{n_I}{n_D} \right\rfloor$ or $\left\lfloor \frac{n_I}{n_D} \right\rfloor + 1$ investors. Then, a

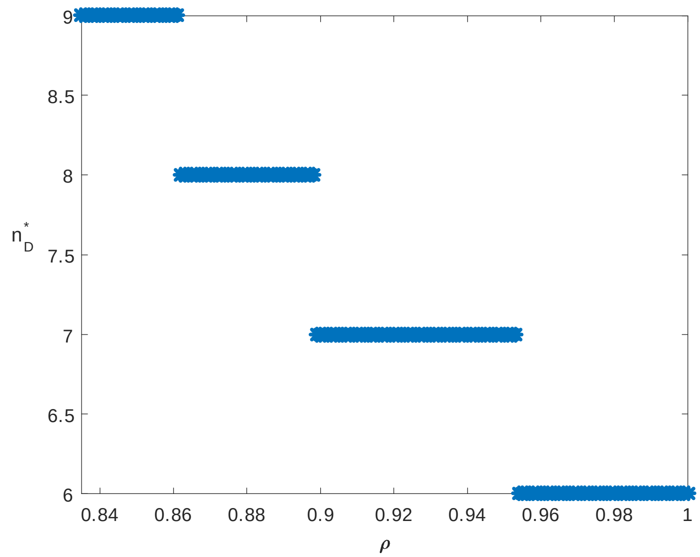

Figure 7: **Dealer Entry.** Equilibrium number of active dealers as a function of ρ for $n_I = 100$, $\gamma = 1$, $\sigma_\theta^2 = 1$ and $F = 0.44$.

dealers's expected profit in a generalized symmetric market structure with n_I investors and n_D dealers is

$$\hat{V}^D(n_D, n_I) = \frac{x}{n_D} V^\ell\left(\left\lfloor \frac{n_I}{n_D} \right\rfloor; n_I, n_D\right) + \left(\frac{n_D - x}{n_D}\right) V^\ell\left(\left\lfloor \frac{n_I}{n_D} \right\rfloor + 1; n_I, n_D\right)$$

Therefore, the dealer entry condition is given by

$$\hat{V}^D(n_D^*, n_I) > F > \hat{V}^D(n_D^* + 1, n_I).$$

Figure 7 shows the number of dealers n_D^* that enters the dealer market, as a function of the correlation between investors' valuations. Each generalized symmetric market structure with n_D^* dealers implied by Figure 7 is supported in equilibrium as introduced in Definition 1. As Figure 7 shows, conditional on fragmentation being an equilibrium, the degree of market fragmentation decreases with the dispersion in investor valuations ρ. From Lemma 3 we know that dealer welfare is decreasing in ρ since a lower dispersion in investor valuations leads to lower gains from intermediation. Therefore, as ρ increases providing intermediation services becomes less profitable and fewer dealers enter the market. This results in a lower degree of market fragmentation.

8 Implications

Our model is informative about whether assets that are traded in over-the-counter markets are inherently different than assets that are traded in centralized markets. In particular, our results imply that assets for which investors have correlated valuations are more likely to be traded in fragmented markets. Moreover, conditional on the correlation in investor valuations being low enough to be traded in fragmented markets, the degree of market fragmentation decreases with this correlation. These results suggest that an important feature in determining whether an asset is traded in a fragmented market is the heterogeneity in investor valuations for the asset. One interpretation of this heterogeneity in investor valuations is that investors disagree about the value of the asset. In this context, the dispersion in asset valuations, $(1 - \rho)$, can be interpreted as a measure of disagreement between investors. In the data, a proxy for disagreement between investors can be obtained using the dispersion in analysts' forecast. Our model suggests that assets for which analysts' forecasts are less dispersed are more likely to be traded in fragmented markets and that the degree of fragmentation is decreasing in this dispersion.

Naturally, our model abstracts from regulatory changes, that could have a major impact on how various assets are traded. However, if one thinks of regulatory reforms as a coordination device between market participants, centralization will arise when market fragmentation is inefficient. If this is the case, assets for which investors have very highly correlated valuations could be traded in centralized markets.

9 Conclusion

We develop a model of market formation in which investors have heterogeneous valuations for the asset to study the determinants of asset market fragmentation. When choosing a dealer, investors trade-off the lower price impact and the higher competition for the dealer's liquidity that a larger market offers. When the correlation among investor valuations is high, the increased competition dominates the decrease in price impact, and investors have no incentives to deviate from a fragmented market structure. We find that dealers can benefit from trading in fragmented asset markets, while investors are always better off in centralized ones. When the correlation in investor valuations is high enough, equilibrium fragmented markets are inefficient. Fragmented markets contribute to lower trading volume relative to a centralized markets.

There are several important features that differentiate markets with different degrees of fragmentation. Some of these characteristics are the liquidity in the market, the trading protocol and the aggregation of information. In this paper, we focus on the liquidity dimension by looking at the price impact of the market participants and the amount of intermediation that dealers are willing to offer. Looking at these features allows us to highlight some essential forces behind the determination of market structure, which are present even in the absence of differences in pricing schedules and informational asymmetries.

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A Appendix: Proofs of the Main Results

Proof of Proposition 1

The first order condition (12) for the dealer's optimization problem yields

$$Q_2^\ell\left(q_1^\ell, p_2; \theta, p_1^\ell\right) = \frac{-\gamma q_1^\ell - p_2}{\gamma + \lambda_2^\ell}.$$

We conjecture and subsequently verify that each dealer's equilibrium strategy is a linear demand function as follows

$$Q_2^\ell\left(q_1^\ell, p_2; \theta, p_1^\ell\right) = a_D q_1^\ell + b_D p_2.$$

Market clearing implies

$$p_2 = -\frac{a_D \sum\limits_{\ell=1}^{N_D} q_1^\ell}{n_D b_D} \text{ and } \lambda_2^\ell = -\frac{1}{\left(n_D - 1\right) b_D}.$$

Then, matching coefficients

$$a_D = -\frac{\gamma}{\gamma - \frac{1}{(n_D-1)b_D}}, \text{ and } b_D = -\frac{1}{\gamma - \frac{1}{(n_D-1)b_D}}. \tag{A.1}$$

The solution for the system in A.1 is given by

$$a_D = -\frac{n_D - 2}{n_D - 1}, \text{ and } b_D = -\frac{1}{\gamma}\frac{n_D - 2}{n_D - 1}.$$

Therefore, the equilibrium strategy of a dealer in the inter-dealer market is

$$Q_2^\ell\left(q_1^\ell, p_2; \theta, p_1^\ell\right) = -\frac{n_D - 2}{n_D - 1}\left(q_1^\ell + \frac{1}{\gamma}p_2\right)$$

and, using the market clearing condition, the equilibrium price is

$$p_2\left(\left\{q_1^l\right\}_{l \in N_D}\right) = -\gamma\frac{\sum_{l \in N_D} q_1^l}{n_D}.$$

Proof of Proposition 2

We conjecture and subsequently verify that in each local market ℓ the dealer's equilibrium strategy is a linear demand function given by

$$Q_1^\ell \left(p_1^\ell;\theta\right) = a^\ell\theta + b^\ell p_1^\ell,$$

and each investor's equilibrium strategy is a linear demand function given by

$$X_1^i \left(p_1^\ell;\theta^i\right) = \alpha^\ell\theta^i + \beta^\ell p_1^\ell.$$

Market clearing implies

$$p_1^\ell = -\frac{\left(n^\ell\alpha^\ell + a^\ell\right)\theta + \alpha^\ell \sum_{i\in N_I(\ell)} \eta^i}{n^\ell\beta^\ell + b^\ell}$$

and

$$\lambda_1^\ell = -\frac{1}{n^\ell\beta^\ell}, \text{ and } \upsilon_1^\ell = -\frac{1}{\left(n^\ell - 1\right)\beta^\ell + b^\ell}.$$

The first order condition (16) for the investor's optimization problem yields

$$x_1^i = \frac{\theta^i - p_1^\ell}{\gamma + \upsilon_1^\ell}. \tag{A.2}$$

The first order condition for the dealer's optimization problem is

$$\frac{dV_2^\ell\left(m\right)}{dq_1^\ell} - p_1^\ell - \frac{\partial p_{1,-\ell}^\ell}{\partial q_1^\ell} q_1^\ell = 0,$$

where $V_2^\ell\left(m\right)$ represents the payoff that dealer ℓ expects at date 1 to receive in the inter-dealer market

$$V_2^\ell\left(m\right) = \mathbb{E}\left[-\frac{\gamma}{2}\left(q_2^{\ell*} + q_1^\ell\right)^2 - p_2 q_2^{\ell*} | \theta, p_1^\ell\right].$$

Differentiating the profits of the dealer in the inter-dealer market, we obtain

$$\frac{dV_2^\ell\left(m\right)}{dq_1^\ell} = \frac{\partial V_2^\ell\left(m\right)}{\partial q_1^\ell} + \frac{\partial V_2^\ell\left(m\right)}{\partial q_2^\ell}\frac{dq_2^\ell}{dq_1^\ell}.$$

Using the first order condition for the dealer in the inter-dealer market we have that

$$\frac{\partial V_2^\ell\left(m\right)}{\partial q_2^\ell} = 0$$

which immediately implies that

$$\frac{dV_2^\ell(m)}{dq_1^\ell} = -\gamma\left(q_1^\ell + \mathbb{E}\left[Q_2^\ell\left(p_2;\theta,q_1^\ell\right)|\theta,q_1^\ell\right]\right).$$

Thus, we can rewrite the first order condition for the dealer as

$$-\gamma\left(q_1^\ell + \mathbb{E}\left[Q_2^\ell\left(p_2;\theta,q_1^\ell\right)|\theta,q_1^\ell\right]\right) - p_1^\ell - \frac{\partial p_{1,-\ell}^\ell}{\partial q_1^\ell} q_1^\ell = 0.$$

Substituting the equilibrium demand function of the dealer in the inter-dealer market, $Q_2^\ell\left(p_2;\theta,q_1^\ell\right)$, we obtain that

$$q_1^\ell = \frac{\left(\gamma+\lambda_2^\ell\right)}{\gamma\lambda_2^\ell + \lambda_1^\ell\left(\gamma+\lambda_2^\ell\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^\ell\right)}\mathbb{E}\left[p_2|\theta,p_1^\ell\right] - p_1^\ell\right).$$

It follows that

$$q_1^\ell = \frac{-\gamma\frac{(n_D-2)}{n_D}\frac{\mathbb{E}\left[\sum_{l\in N_D,l\neq\ell} q_1^l|\theta\right]}{n_D-1} - p_1^\ell}{\gamma\frac{2}{n_D} + \frac{\partial p_1^\ell}{\partial q_1^\ell}}, \tag{A.3}$$

where

$$\begin{aligned}
\mathbb{E}\left[\sum_{l\in N_D,l\neq\ell} q_1^l|\theta,q_1^\ell\right] &= \mathbb{E}\left[\sum_{l\in N_D,l\neq\ell}\left(a^h\theta - b^h\left(\frac{\left(n^h\alpha^h + a^h\right)\theta + \alpha^h\sum_{i\in N_I(h)}\eta^i}{n^h\beta^h + b^h}\right)\right)|\theta\right] \\
&= \sum_{h\in N_D,h\neq\ell}\left(a^h - b^h\frac{\left(n^h\alpha^h + a^h\right)}{n^h\beta^h + b^h}\right)\theta = \sum_{h\in N_D,h\neq\ell} n^h\frac{a^h\beta^h - b^h\alpha^h}{n^h\beta^h + b^h}\theta.
\end{aligned}$$

Using the demand functions in equations $(A.3)$ and $(A.2)$, and matching the coefficients with our guess for an equilibrium in linear strategies we get the following system

$$\begin{aligned}
a^\ell &= b^\ell\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\sum_{l\in N_D,l\neq\ell} n^l\frac{a^l\beta^l - b^l\alpha^l}{n^l\beta^l + b^l} \\
b^\ell &= -\frac{n^\ell\beta^\ell n_D}{2\gamma n^\ell\beta^\ell - n_D}, \\
\alpha^\ell &= \frac{\left(n^\ell - 1\right)\beta^\ell + b^\ell}{\gamma\left(\left(n^\ell-1\right)\beta^\ell + b^\ell\right) - 1}, \text{ and} \\
\beta^\ell &= -\frac{\left(n^\ell - 1\right)\beta^\ell + b^\ell}{\gamma\left(\left(n^\ell-1\right)\beta^\ell + b^\ell\right) - 1}.
\end{aligned} \tag{A.4}$$

We solve for β^ℓ in the system of equations $(A.4)$. It follows that β^ℓ is given by the negative solution to

$$H(\beta) = 0,$$

where

$$H(\beta) = -2n^\ell \left(n^\ell - 1\right)(\gamma\beta)^2 + \left(\left(2n^\ell - 1\right) n_D + 2n^\ell - 2n^\ell \left(n^\ell - 1\right)\right)\gamma\beta + 2\left(n^\ell - 1\right) n_D.$$

Since $H(0) > 0$ and $H''(\cdot) < 0$, a solution to $H(\beta) = 0$ always exists and there is a unique negative root which determines β^ℓ. Then, $\left\{\alpha^\ell, \beta^\ell, b^\ell\right\}_{\ell \in N_D}$ are uniquely determined. Let $\vec{a} = [a^1, a^2, ..., a^{n_D}]'$. From $(A.4)$ we have

$$\vec{a} = A\vec{a} + B,$$

where A is a $n_D \times n_D$ matrix where the element $A_{\ell l} = b^\ell \gamma \frac{(n_D-2)}{n_D} \frac{1}{n_D-1} \frac{n^l \beta^l}{n^l \beta^l + b^l} \in (-1, 0)$ for all $l \neq \ell$, $l, \ell \in N_D$ and $A_{\ell\ell} = 0$ for all $\ell \in N_D$; and B is a $n_D \times 1$ matrix where the element

$$B_\ell = -b^\ell \gamma \frac{(n_D - 2)}{n_D} \frac{1}{n_D - 1} \sum_{l \in N_D, l \neq \ell} n^l \frac{b^l \alpha^l}{n^l \beta^l + b^l} \text{ for all } \ell \in N_D.$$

Then,

$$\left[I_{n_D} - A\right] \vec{a} = B,$$

where $[I_{n_D} - A]$ is invertible. Therefore, $\vec{a}$ is uniquely determined.

Proof of Lemma 1

Consider a market ℓ. The first order condition (16) for the investor's optimization problem yields

$$X_1^i\left(p_1^\ell; \theta^i\right) = \frac{\theta^i - p_1^\ell}{\gamma + \upsilon_1^\ell}.$$

As we show in the proof of Proposition 2, the first order condition (17) for the dealer's optimization problem implies that

$$Q_1^\ell\left(p_1^\ell; \theta\right) = \frac{\left(\gamma + \lambda_2^\ell\right)}{\gamma\lambda_2^\ell + \lambda_1^\ell\left(\gamma + \lambda_2^\ell\right)} \left(\frac{\gamma}{\left(\gamma + \lambda_2^\ell\right)} \mathbb{E}\left[p_2 | \theta, p_1^\ell\right] - p_1^\ell\right). \tag{A.5}$$

The market clearing condition (5) becomes

$$\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^\ell\right)}\mathbb{E}\left[p_2|\theta,p_1^\ell\right]-p_1^\ell\right)-\frac{1}{\upsilon_1^\ell+\gamma}\sum_{i\in N_I(\ell)}\left(\theta^i-p_1^\ell\right)=0,$$

which implies that the price in the local market ℓ is

$$p_1^\ell=\frac{1}{\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)}-\frac{n^\ell}{\upsilon_1^\ell+\gamma}}\left(\frac{\gamma}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)}\mathbb{E}\left(p_2|\theta,p_1^\ell\right)-\frac{n^\ell}{\upsilon_1^\ell+\gamma}\frac{\sum_{i\in N_I(\ell)}\theta^i}{n^\ell}\right).$$

Further, the price in the inter-dealer market is

$$p_2=-\gamma\frac{\sum_{l\in N_D}q_1^l}{n_D}$$

and, substituting (A.5), we obtain

$$p_2=-\frac{\gamma}{n_D}\sum_{l\in N_D}\frac{\left(\gamma+\lambda_2^l\right)}{\left(\gamma\lambda_2^l+\gamma\lambda_1^l+\lambda_1^l\lambda_2^l\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left(p_2|\theta,p_1^l\right)-p_1^l\right).$$

Taking expectations, we have that

$$\mathbb{E}\left(p_2|\theta\right)=-\frac{\gamma}{n_D}\sum_{l\in N_D}\frac{\left(\gamma+\lambda_2^l\right)}{\left(\gamma\lambda_2^l+\gamma\lambda_1^l+\lambda_1^l\lambda_2^l\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left(\mathbb{E}\left(p_2|\theta,p_1^l\right)|\theta\right)-\mathbb{E}\left(p_1^l|\theta\right)\right),$$

and

$$\mathbb{E}\left(p_1^l|\theta\right)=\frac{1}{\frac{\left(\gamma+\lambda_2^l\right)}{\left(\gamma\lambda_2^l+\gamma\lambda_1^l+\lambda_1^l\lambda_2^l\right)}-\frac{n^l}{\upsilon_1^l+\gamma}}\left(\frac{\gamma}{\left(\gamma\lambda_2^l+\gamma\lambda_1^l+\lambda_1^l\lambda_2^l\right)}\mathbb{E}\left(\mathbb{E}\left(p_2|\theta,p_1^l\right)|\theta\right)-\frac{n^l}{\upsilon_1^l+\gamma}\theta\right).$$

It follows that

$$\frac{\left(\gamma+\lambda_2^l\right)}{\left(\gamma\lambda_2^l+\gamma\lambda_1^l+\lambda_1^l\lambda_2^l\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left(\mathbb{E}\left(p_2|\theta,p_1^l\right)|\theta\right)-\mathbb{E}\left(p_1^l|\theta\right)\right)=-\frac{n^l}{\upsilon_1^l+\gamma}\left(\mathbb{E}\left(p_1^l|\theta\right)-\theta\right),\qquad\text{(A.6)}$$

which implies that

$$\mathbb{E}\left(p_2|\theta\right)=\frac{\gamma}{n_D}\sum_{l\in N_D}\frac{n^l}{\upsilon_1^l+\gamma}\left(\mathbb{E}\left(p_1^l|\theta\right)-\theta\right).$$

Further, using that $\mathbb{E}\left(\mathbb{E}\left(p_2|\theta,p_1^l\right)|\theta,p_1^\ell\right)=\mathbb{E}\left(\mathbb{E}\left(p_2|\theta,p_1^l\right)|\theta\right)$, we obtain

$$\begin{aligned}\mathbb{E}\left(p_2|\theta,p_1^\ell\right) &= -\frac{\gamma}{n_D}\sum_{\substack{l\in N_D\\ l\neq\ell}}\frac{\left(\gamma+\lambda_2^l\right)}{\left(\gamma\lambda_2^l+\gamma\lambda_1^l+\lambda_1^l\lambda_2^l\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^l\right)}\mathbb{E}\left(\mathbb{E}\left(p_2|\theta,p_1^l\right)|\theta\right)-\mathbb{E}\left(p_1^l|\theta\right)\right)\\ &\quad -\frac{\gamma}{n_D}\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^\ell\right)}\mathbb{E}\left(p_2|\theta,p_1^\ell\right)-p_1^\ell\right).\end{aligned}$$

Using (A.6), we obtain that

$$\mathbb{E}\left(p_2|\theta,p_1^\ell\right)=\frac{\gamma}{n_D}\sum_{\substack{l\in N_D\\ l\neq\ell}}\frac{n^l}{\upsilon_1^l+\gamma}\left(\mathbb{E}\left(p_1^l|\theta\right)-\theta\right)-\frac{\gamma}{n_D}\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)}\left(\frac{\gamma}{\left(\gamma+\lambda_2^\ell\right)}\mathbb{E}\left(p_2|\theta,p_1^\ell\right)-p_1^\ell\right).$$

This gives us that

$$\mathbb{E}\left(p_2|\theta,p_1^\ell\right)=\frac{\frac{\gamma}{n_D}}{\left(1+\gamma\frac{1}{n_D}\frac{\gamma}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)}\right)}\sum_{\substack{l\in N_D\\ l\neq\ell}}\frac{n^l}{\upsilon_1^l+\gamma}\left(\mathbb{E}\left(p_1^l|\theta\right)-\theta\right)+\frac{\frac{\gamma}{n_D}\left(\gamma+\lambda_2^\ell\right)}{\left(\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\gamma\frac{1}{n_D}\gamma\right)}p_1^\ell.$$

Substituting back into (A.5), we obtain that the demand function of a dealer in the local market is

$$Q_1^\ell\left(p_1^\ell;\theta\right)=\frac{\frac{\gamma^2}{n_D}}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}\sum_{\substack{l\in N_D\\ l\neq\ell}}\frac{n^l}{\upsilon_1^l+\gamma}\left(\mathbb{E}\left(p_1^l|\theta\right)-\theta\right)-\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}p_1^\ell.$$

Using again market clearing and identifying coefficients we obtain that

$$\upsilon_1^\ell=\frac{1}{\frac{n^\ell-1}{\gamma+\upsilon_1^\ell}+\frac{\left(\gamma+\lambda_2^\ell\right)}{\left(\gamma\lambda_2^\ell+\gamma\lambda_1^\ell+\lambda_1^\ell\lambda_2^\ell\right)+\frac{\gamma^2}{n_D}}}$$

and

$$\lambda_1^\ell=\frac{1}{\frac{n^\ell}{\gamma+\upsilon_1^\ell}}.$$

Proof of Theorem 1

We prove the result in three steps. First, we show in Lemma $A.1$ below that

$$\lim_{n_D\to\infty}\Delta^i\left(\rho=1;n_D,n_S\right)<0,\tag{A.7}$$

where Δ^i has been defined in Eq.(26). Second, we show in Lemma $A.2$ below that

$$\lim_{n_D\to 3}\Delta^i\left(\rho=1;n_D,n_S\right)>0.\tag{A.8}$$

Lastly, we show in Lemma $A.3$ $\Delta\left(\rho;n_D,n_S\right)$ is monotonically increasing in ρ, or

$$\frac{\partial\Delta^i\left(\rho;n_D,n_S\right)}{\partial\rho}>0.\tag{A.9}$$

From (A.8) and (A.9) it follows that $\exists\bar{\rho}$ such that for all $\rho > \bar{\rho}$, $\Delta(\rho; n_D = 3) > 0$. From (A.7) and (A.9) it follows that for any ρ

$$\lim_{n_D \to \infty} \Delta^i(\rho; n_D, n_S) < 0,$$

and, in particular, for all $\rho > \bar{\rho}$. Therefore, since $\Delta^i(\rho; n_D, n_S)$ is continuous in n_D, for all $\rho^* > \bar{\rho}$ there exists there exists $n_D(\rho^*)$ such that

$$\Delta^i(\rho^*; n_D(\rho^*), n_S) = 0.$$

Using (A.9), this implies that for all $\rho > \rho^*$

$$\Delta^i(\rho; n_D(\rho^*), n_S) > 0.$$

Next, we present the proofs of the aiding lemmas. For the remainder of the appendix, $\beta(n, n_D)$ is the negative root of $H(\beta; n, n_D) = 0$. Similarly, $b(n, n_D)$ satisfies the system (A.4).

Lemma A.1 *If investor valuations are perfectly correlated, an investor has incentives to deviate if the inter-dealer market is competitive, i.e.,*

$$\lim_{n_D \to \infty} \Delta^i(\rho = 1; n_D, n_S) < 0.$$

Proof. From the definition of Δ^i in Eq.(26) we have

$$\Delta^i(\rho = 1; n_D, n_S) = -\left(\frac{\beta^{sym}\left(1 + \frac{\gamma}{2}\beta^{sym}\right)}{\left(n_S\beta^{sym} + b^{sym}\right)^2}\left(b^{sym} + a^{sym}\right)^2 - \frac{\beta^{dev}\left(1 + \frac{\gamma}{2}\beta^{dev}\right)}{\left((n_S+1)\beta^{dev} + b^{dev}\right)^2}\left(b^{dev} + a^{dev}\right)^2\right)\sigma_\theta^2,$$

where

$$\beta^{sym} = \beta(n_S, n_D) \text{ and } \beta^{dev} = \beta(n_S + 1, n_D),$$

and a^{sym} and a^{dev} are as defined in equations $(A.4)$ and $(C.2)$, respectively. Then,

$$\lim_{n_D \to \infty} \Delta^i(1; n_D, n_S) = -\left(\frac{\bar{\beta}^{sym}\left(1 + \frac{\gamma}{2}\bar{\beta}^{sym}\right)\left(n_S\bar{\beta}^{sym} + \lim_{n_D \to \infty} a^{sym}\right)^2}{\left(2n_S\bar{\beta}^{sym}\right)^2} - \frac{\beta^{dev}\left(1 + \frac{\gamma}{2}\beta^{dev}\right)\left((n_S+1)\bar{\beta}^{dev} + \lim_{n_D \to \infty} a^{dev}\right)^2}{\left(2(n_S+1)\bar{\beta}^{dev}\right)^2}\right)\sigma_\theta^2.$$

Using Lemma $B.10$ and Lemma $C.13$ we have

$$\lim_{n_D \to \infty} \Delta^i(1; n_D, n_S) = -\left(\frac{1}{2 - \gamma\bar{\beta}^{sym} n_S}\right)^2 \frac{4n_S}{\gamma\left(4n_S^2 - 1\right)^2}\sigma_\theta^2 < 0$$

where we used that

$$\bar{\beta}^{sym} \equiv \lim_{n_D \to \infty} \beta(n_S) = -\frac{2\left(n_S - 1\right)}{\gamma\left(2n_S - 1\right)}, \text{ and } \bar{\beta}^{dev} \equiv \lim_{n_D \to \infty} \beta\left(n_S + 1\right) = -\frac{2n_S}{\gamma\left(2n_S + 1\right)}.$$

■

Lemma A.2 *If investor valuations are perfectly correlated, an investor has no incentives to deviate if there are three active dealers, i.e.,*

$$\lim_{n_D \to 3} \Delta^i\left(\rho = 1; n_D, n_S\right) > 0.$$

Proof. Let $b^{sym} \equiv b\left(n_S, n_D = 3\right)$, $b^{dev} \equiv b\left(n_S + 1, n_D = 3\right)$ and $b^{\ell} \equiv b\left(n_S - 1, n_D = 3\right)$. From the definition of Δ^i in Eq.(26) we have

$$\begin{aligned}
& \lim_{n_D \to 3} \Delta^i\left(\rho = 1; n_D, n_S\right) \\
= & \lim_{n_D \to 3} -\left(\frac{\beta^{sym}\left(1 + \frac{\gamma}{2}\beta^{sym}\right)}{\left(n_S\beta^{sym} + b^{sym}\right)^2}\left(b^{sym} + a^{sym}\right)^2 - \frac{\beta^{dev}\left(1 + \frac{\gamma}{2}\beta^{dev}\right)}{\left(\left(n_S + 1\right)\beta^{dev} + b^{dev}\right)^2}\left(b^{dev} + a^{dev}\right)^2\right)\sigma_\theta^2 \\
= & \lim_{n_D \to 3} -\left(\frac{\beta^{sym}\left(1 + \frac{\gamma}{2}\beta^{sym}\right)}{\left(n_S\beta^{sym} + b^{sym}\right)^2}\left(b^{sym} + a^{sym}\right)^2 - \frac{\beta^{dev}\left(1 + \frac{\gamma}{2}\beta^{dev}\right)}{\left(\left(n_S + 1\right)\beta^{dev} + b^{dev}\right)^2}\left(b^{dev} + a^{dev}\right)^2\right)\sigma_\theta^2 \\
= & \lim_{n_D \to 3} -\left(\begin{array}{c} \frac{\frac{3b^{sym}}{n_S(2\gamma b^{sym}+3)}\left(1+\frac{\gamma}{2}\frac{3b^{sym}}{n_S(2\gamma b^{sym}+3)}\right)}{\left(\frac{3b^{sym}}{(2\gamma b^{sym}+3)}+b^{sym}\right)^2}\left(2b^{sym}\frac{\gamma b^{sym}+3}{\gamma b^{sym}+6}\right)^2 \\ -\frac{\beta^{dev}\left(1+\frac{\gamma}{2}\beta^{dev}\right)}{\left((n_S+1)\beta^{dev}+b^{dev}\right)^2}\left(b^{dev} + a^{dev}\right)^2 \end{array}\right)\sigma_\theta^2 \\
= & \lim_{n_D \to 3} -\left(\begin{array}{c} \frac{3}{2}\frac{b^{sym}(2n_S(2\gamma b^{sym}+3)+3\gamma b^{sym})}{(n^{\ell})^2(\gamma b^{sym}+6)^2} \\ -\frac{3}{2}\frac{b^{dev}\left(2\left(2\gamma b^{dev}+3\right)(n_S+1)+\gamma 3b^{dev}\right)}{(n_S+1)^2(\gamma b^{dev}+6)^2}\left(\frac{\left(\gamma b^{dev}+6\right)}{(2\gamma b^{dev}+6)}\left(1 + \frac{a^{dev}}{b^{dev}}\right)\right)^2 \end{array}\right)\sigma_\theta^2,
\end{aligned}$$

where we used that

$$a^{sym} = \frac{\gamma\left(b^{sym}\right)^2}{\gamma b^{sym} + 6} \text{ and } n^{\ell}\beta^{sym} = \frac{3b^{sym}}{\left(2\gamma b^{sym} + 3\right)}.$$

From Lemma $C.14$ in the Online Appendix we know that

$$-\frac{\left(4\gamma b^{sym}n_S+6n_S+\gamma 3b^{sym}\right)}{n_S\left(\gamma b^{sym}+6\right)^2}b^{sym} + \frac{\left(4\gamma(n_S+1)b^{dev}+6(n_S+1)+\gamma 3b^{dev}\right)}{(n_S+1)\left(\gamma b^{dev}+6\right)^2}b_{dev} > 0 \tag{A.10}$$

and from Lemma $C.15$ in the Online Appendix we have

$$\frac{n_S + 1}{n_S} > \left(\frac{\left(\gamma b^{dev} + 6\right)}{\left(2\gamma b^{dev} + 6\right)}\left(1 + \frac{a^{dev}}{b^{dev}}\right)\right)^2 > 0. \tag{A.11}$$

Then, from Eq.$(A.10)$ we have

$$-\frac{\left(4\gamma b^{sym}n_S+6n_S+\gamma 3b^{sym}\right)}{n_S^2\left(\gamma b^{sym}+6\right)^2}b^{sym}\left(n_S+1\right)+\frac{\left(4\gamma\left(n_S+1\right)b^{dev}+6\left(n_S+1\right)+\gamma 3b^{dev}\right)}{n_S\left(\gamma b^{dev}+6\right)^2}b_{dev}>0$$

and using Eq.$(A.11)$ it follows that

$$\lim_{n_D\to 3}\Delta^i\left(\rho=1;n_D,n_S\right)>0.$$

■

Lemma A.3 *Δ^i is monotonically increasing in ρ, i.e., $\frac{\partial\Delta^i(\rho;n_D,n_S)}{\partial\rho}>0$.*

Proof. From the definition of Δ^i we have

$$\frac{\partial\Delta^i\left(\rho;n_D,n_S\right)}{\partial\rho}=L\left(n_S+1\right)-L\left(n_S\right),$$

where

$$\begin{aligned}L\left(n\right) &\equiv -\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\left(\left(\frac{\left(b\left(n\right)+\left(n-1\right)\beta\left(n\right)\right)}{\left(b\left(n\right)+n\beta\left(n\right)\right)}\right)^2+\left(n-1\right)\left(\frac{\beta\left(n\right)}{\left(b\left(n\right)+n\beta\left(n\right)\right)}\right)^2\right)\\ &= -\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\left(1-\frac{\beta\left(n\right)}{\left(b\left(n\right)+n\beta\left(n\right)\right)}-\frac{\beta\left(n\right)b\left(n\right)}{\left(b\left(n\right)+n\beta\left(n\right)\right)^2}\right)>0.\end{aligned}$$

Taking the derivative with respect to n we have

$$\frac{\partial L\left(n\right)}{\partial n}=\frac{d}{dn}\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\left(1-\frac{\beta\left(n\right)}{\left(b\left(n\right)+n\beta\left(n\right)\right)}-\frac{\beta\left(n\right)b\left(n\right)}{\left(b\left(n\right)+n\beta\left(n\right)\right)^2}\right)\right)>0,$$

since from Lemma $C.16$ in the Online Appendix $1-\frac{\beta(n)}{(b(n)+n\beta(n))}$ is increasing in n, from Lemma $C.18$ we know $-\frac{\beta(n)b(n)}{(b(n)+n\beta(n))^2}$ is increasing in n and

$$\frac{\partial\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\right)}{\partial n}=-\left(\beta\left(n\right)\gamma+1\right)\frac{\partial\beta\left(n\right)}{\partial n}>0,$$

because $0>\beta\left(n\right)>-\frac{1}{\gamma}$ and $\frac{\partial\beta}{\partial n}<0$. Then, $L\left(n_S+1\right)-L\left(n_S\right)>0$. ■

Proof of Proposition 3

The proof follows from Lemma $A.1$ which shows that

$$\lim_{n_D \to \infty} \Delta^i (\rho = 1; n_D, n_S) < 0,$$

and Lemma $A.3$ which shows that $\Delta^i (\rho; n_D, n_S)$ is monotonically increasing in ρ. This implies that

$$\Delta^i (\rho; n_D, n_S) < \Delta^i (\rho = 1; n_D, n_S)$$

for any $\rho < 1$. Taking the limit of $n_D \to \infty$, we obtain that

$$\lim_{n_D \to \infty} \Delta^i (\rho; n_D, n_S) < 0.$$

Proof of Lemma 2

The expected utility of an investor i in local market ℓ in a symmetric market structure is

$$V_1^i (m_{n_S}; \rho) = \mathbb{E}\left[\theta^i x_1^i - \frac{\gamma}{2}\left(x_1^i\right)^2 - p_1^\ell x_1^i\right], \tag{A.12}$$

where x_1^i and p_1^ℓ are the equilibrium quantity acquired by investor i and the equilibrium price in local market ℓ, respectively. In this case $n^\ell = n_S \; \forall \ell \in N_D$.

The equilibrium price in local market ℓ is

$$p_1^\ell = \pi_\theta \theta + \pi_\eta \sum_{i \in N^\ell} \eta^i,$$

where

$$\pi_\theta = -\frac{\left(n^\ell \alpha^\ell + a^\ell\right)}{n^\ell \beta^\ell + b^\ell} \text{ and } \pi_\eta = \frac{\beta^\ell}{n^\ell \beta^\ell + b^\ell}.$$

Using the equilibrium linear strategies, we have

$$\mathbb{E}\left[\theta^i x_1^i\right] = \mathbb{E}\left[\theta^i \left(\alpha^\ell \theta^i + \beta^\ell p_1^\ell\right)\right] = -\frac{\beta^\ell \left(a^\ell + b^\ell\right)}{b^\ell + n^\ell \beta^\ell}\sigma_\theta^2 - \frac{\left(n^\ell - 1\right)\beta^\ell + b^\ell}{n^\ell \beta^\ell + b^\ell}\beta^\ell \sigma_\eta^2,$$

$$\begin{aligned}
\mathbb{E}\left[\left(x_1^i\right)^2\right] &= \mathbb{E}\left[\left(\alpha^\ell \theta^i + \beta^\ell p_1^\ell\right)^2\right] \\
&= \left(\frac{\beta^\ell}{b^\ell + n^\ell \beta^\ell}\left(a^\ell + b^\ell\right)\right)^2 \sigma_\theta^2 + \left(\left(\left(n^\ell - 1\right)\beta^\ell + b^\ell\right)^2 + \left(n^\ell - 1\right)\beta^{\ell 2}\right)\left(\frac{\beta^\ell}{n^\ell \beta^\ell + b^\ell}\right)^2 \sigma_\eta^2,
\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[p_1^\ell x_1^i\right] &= \pi_\theta\left(\alpha^\ell+\beta^\ell\pi_\theta\right)\sigma_\theta^2+\pi_\eta\left(\alpha^\ell+\beta^\ell\pi_\eta\right)\sigma_\eta^2+\beta^\ell\pi_\eta^2\left(n^\ell-1\right)\sigma_\eta^2\\ &= \frac{\left(-n^\ell\beta+a^\ell\right)}{\left(n^\ell\beta^\ell+b^\ell\right)^2}\beta^\ell\left(a^\ell+b^\ell\right)\sigma_\theta^2+\left(\frac{\beta^\ell}{n^\ell\beta^\ell+b^\ell}\right)^2\left(b^\ell+2n^\ell\beta^\ell\right)\sigma_\eta^2.\end{aligned}$$

Then, Eq.(A.12) becomes

$$V_1^i\left(m_n;\rho\right)=-\beta^\ell\left(\frac{\gamma}{2}\beta^\ell+1\right)\frac{1}{\left(n^\ell\beta^\ell+b^\ell\right)^2}\left(\left(a^\ell+b^\ell\right)^2+\left(\left(\left(n^\ell-1\right)\beta^\ell+b^\ell\right)^2+\left(n^\ell-1\right)\beta^{\ell 2}\right)\frac{1-\rho}{\rho}\right)\sigma_\theta^2. \tag{A.13}$$

Since and β^ℓ and b^ℓ do not depend on ρ, and $\beta^\ell\in\left(-\frac{1}{\gamma},0\right)$, we have that

$$\frac{\partial V_1^i\left(m;\rho\right)}{\partial\rho}=\beta^\ell\left(\frac{\gamma}{2}\beta^\ell+1\right)\frac{\left(\left(\left(n^\ell-1\right)\beta^\ell+b^\ell\right)^2+\left(n^\ell-1\right)\beta^{\ell 2}\right)}{\left(n^\ell\beta^\ell+b^\ell\right)^2}\frac{1}{\rho^2}<0.$$

Proof of Proposition 5

Investor welfare can be rewritten as

$$V_1^i\left(m;\rho\right)=-\beta^\ell\left(\frac{\gamma}{2}\beta^\ell+1\right)\left(\left(1-\pi_\theta\right)^2+\left(\left(1-\pi_\eta\right)^2+\pi_\eta^2\left(n^\ell-1\right)\right)\frac{1-\rho}{\rho}\right)\sigma_\theta^2 \tag{A.14}$$

where π_θ and π_η are such that

$$p_1=\pi_\theta\theta+\pi_\eta\sum_{j\in N^\ell}\eta^j$$

Let $\Delta V^i=\mathbb{E}\left(V^{i,c}\right)-\mathbb{E}\left(V^i\right)$ be the difference between an investor's welfare in a centralized market with n_D dealers and a symmetric fragmented market with n_D dealers. The proof follows directly from Lemma A.4 and Lemma A.5.

Lemma A.4 *Investors are better off in a centralized market structure as ρ approaches 1, i.e.,*

$$\lim_{\rho\to 1}\Delta V^i=-\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\left(\frac{n_D}{n_I+n_D}\right)^2-\left(-\beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\right)\left(-\frac{1}{\beta^s\gamma n_S-2}\right)^2>0$$

Proof. We can write

$$\lim_{\rho\to 1}\Delta V^i = \left(\beta^s\left(\frac{\gamma}{2}\beta^s+1\right)-\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\right)\left(-\frac{1}{\beta^s\gamma n_S-2}\right)^2-\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\left(\left(\frac{1}{n_S+1}\right)^2-\left(\frac{1}{2-\beta^s\gamma n_S}\right)^2\right) \tag{A.15}$$

which implies $\lim_{\rho\to 1}\Delta V^i > 0$ since

$$\frac{\partial\left(\beta\left(\frac{\gamma}{2}\beta+1\right)\right)}{\partial\beta} = \beta\gamma+1>0$$

and from Lemma B.11 we have $\beta^s > \beta^c$, the first term in Eq. $A.15$ is positive. Also,

$$\begin{aligned}\frac{1}{n_S+1} &> \frac{1}{2-\beta^s\gamma n_S} > 0\\ \beta^s\gamma &< -\frac{n_S-1}{n_S}\end{aligned}$$

since $H\left(-\frac{n_S-1}{\gamma n_S}\right) = \frac{1}{n_S}(n_S-1)(n_D-2) > 0$ and the second term is positive, which implies

$$\lim_{\rho\to 1}\Delta V^i > 0.$$

■

Lemma A.5 ΔV^i *is monotonically decreasing in* ρ

$$\frac{\partial\Delta V^i}{\partial\rho} = \left(\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\left(\left(1-\pi^c_\eta\right)^2+\pi^{c2}_\eta(n_I-1)\right)-\beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\left(\left(1-\pi^s_\eta\right)^2+\pi^{s2}_\eta(n_S-1)\right)\right)\frac{1}{\rho^2}\sigma^2_\theta<0. \tag{A.16}$$

Proof. Since β^c, β^s, π^c_η and π^s_η are independent of ρ, monotonicity follows from Eq. $A.16$. Note that

$$sign\left[\frac{\partial\Delta V^i}{\partial\rho}\right] = sign\left[\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)\left(\left(1-\pi^c_\eta\right)^2+\pi^{c2}_\eta(n_I-1)\right)-\beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\left(\left(1-\pi^s_\eta\right)^2+\pi^{s2}_\eta(n_S-1)\right)\right] \tag{A.17}$$

where

$$\pi^s_\eta = \frac{\beta^s}{n_S\beta^s+b^s} = \frac{1}{n_S}\frac{n_D-2\beta^s\gamma n_S}{2n_D-2\beta^s\gamma n_S}$$

and

$$\pi^c_\eta = \frac{1}{n_I+n_D}.$$

Rewriting the right hand side of Eq. $A.17$ we have

$$sign\left[\frac{\partial \Delta V^i}{\partial \rho}\right] = sign\left[\begin{array}{c} \left(\beta^c\left(\frac{\gamma}{2}\beta^c+1\right)-\beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\right)\left(\left(1-\pi_\eta^c\right)^2+\pi_\eta^{c2}\left(n_I-1\right)\right) \\ -\beta^s\left(\frac{\gamma}{2}\beta^s+1\right)\left(\left(1-\pi_\eta^s\right)^2+\pi_\eta^{s2}\left(n_S-1\right)-\left(1-\pi_\eta^c\right)^2-\pi_\eta^{c2}\left(n_I-1\right)\right) \end{array}\right].$$

The first term is negative since $\frac{\partial\left(\beta\left(\frac{\gamma}{2}\beta+1\right)\right)}{\partial\beta} > 0$ and from Lemma B.11 we have $\beta^s > \beta^c$.

From Lemma B.12 we know that $\pi_\eta^s > \pi_\eta^c$. Let

$$\Omega(x) = (1-x)^2 + x^2(n_S-1).$$

Then

$$\Omega'(x) = -2(1-x) + 2x(n_S-1) = 2(xn_S-1).$$

Since $\pi_\eta^s n_S < 1$ and $n_S\pi_\eta^c < 1$ we have that $\Omega'(x) < 0$ for all x in $\left[\pi_\eta^c, \pi_\eta^s\right]$. Since

$$\begin{aligned} & \left(1-\pi_\eta^s\right)^2+\pi_\eta^{s2}\left(n_S-1\right)-\left(1-\pi_\eta^c\right)^2-\pi_\eta^{c2}\left(n_I-1\right)= \\ = & \left(1-\pi_\eta^s\right)^2+\pi_\eta^{s2}\left(n_S-1\right)-\left(\left(1-\pi_\eta^c\right)^2+\pi_\eta^{c2}\left(n_S-1\right)\right)-\pi_\eta^{c2}\left(n_I-n_S\right) \\ = & \left(1-\pi_\eta^s\right)^2+\pi_\eta^{s2}\left(n_S-1\right)-\left(\left(1-\pi_\eta^c\right)^2+\pi_\eta^{c2}\left(n_S-1\right)\right)-\pi_\eta^{c2}n_S\left(n_D-1\right) \\ = & \Omega\left(\pi_\eta^s\right)-\Omega\left(\pi_\eta^c\right)-\pi_\eta^{c2}n_S\left(n_D-1\right)<0 \end{aligned}$$

we have $\frac{\partial\Delta V^i}{\partial\rho} < 0$. ■

Proof of Lemma 3

The expected utility of a dealer ℓ in a symmetric market structure is

$$W_D^\ell = -\mathbb{E}\left[\frac{\gamma}{2}\left(q_1^\ell+q_2^\ell\right)^2+p_1^\ell q_1^\ell+p_2 q_2^\ell\right], \tag{A.18}$$

where q_1^ℓ and q_2^ℓ are the equilibrium quantities acquired by the dealer in the local and inter-dealer markets, respectively, and p_1 and p_2 are the equilibrium prices in these markets.

Using the equilibrium linear strategies, we have

$$\begin{aligned} \mathbb{E}\left[\left(q_1^\ell+q_2^\ell\right)^2\right] &= \mathbb{E}\left[\left(\frac{2}{n_D}q_1^\ell+\frac{(n_D-2)}{n_D}\frac{\sum_{l\in N_D,l\neq\ell}q_1^l}{n_D-1}\right)^2\right] \\ &= \left(a^\ell+b^\ell\pi_\theta\right)^2\sigma_\theta^2+\left(\frac{2}{n_D}b^\ell\pi_\eta\right)^2 n^\ell\sigma_\eta^2+\left(\frac{(n_D-2)}{n_D}b^\ell\pi_\eta\right)^2\frac{n^\ell\sigma_\eta^2}{n_D-1} \\ &= \left(\frac{n^\ell\beta^\ell}{b^\ell+n^\ell\beta^\ell}\left(a^\ell+b^\ell\right)\right)^2\sigma_\theta^2+\left(\frac{\beta^\ell}{n^\ell\beta^\ell+b^\ell}b^\ell\right)^2\frac{n^\ell\sigma_\eta^2}{n_D-1}, \end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[p_1^\ell q_1^\ell\right] &= \mathbb{E}\left[\left(\pi_\theta\theta+\pi_\eta\sum_{i\in N^\ell}\eta^i\right)\left(a^\ell\theta+b^\ell\left(\pi_\theta\theta+\pi_\eta\sum_{i\in N^\ell}\eta^i\right)\right)\right] \\
&= -\frac{n^\ell\beta^\ell}{\left(b^\ell+n^\ell\beta^\ell\right)^2}\left(a^\ell+b^\ell\right)\left(n^\ell\alpha^\ell+a^\ell\right)\sigma_\theta^2+b^\ell\left(\frac{\beta^\ell}{n^\ell\beta^\ell+b^\ell}\right)^2 n^\ell\sigma_\eta^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[p_2 q_2^\ell\right] &= \mathbb{E}\left[\left(\gamma\frac{\sum_{l\in N_D}q_1^l}{n_D}\right)\frac{(n_D-2)}{n_D}\left(\frac{\sum_{l\in N_D,l\neq\ell}q_1^l}{n_D-1}-q_1^\ell\right)\right] \\
&= \gamma\frac{(n_D-2)}{n_D}\left(b^\ell\pi_\eta\right)^2\left(\frac{1}{n_D}n^\ell\sigma_\eta^2-\frac{1}{n_D}n^\ell\sigma_\eta^2\right)=0.
\end{aligned}$$

Then, Eq.$(A.18)$ becomes

$$V_D\left(m_n,\rho\right)=\left(\tfrac{n^\ell\beta^\ell}{b^\ell+n^\ell\beta^\ell}\right)^2\left(\left(-\frac{\gamma}{2}\left(a^\ell+b^\ell\right)+\frac{\left(n^\ell\alpha^\ell+a^\ell\right)}{n^\ell\beta^\ell}\right)\left(a^\ell+b^\ell\right)-\left(1+\frac{\gamma}{2}\frac{1}{n_D-1}b^\ell\right)\frac{b^\ell}{n^\ell}\frac{1-\rho}{\rho}\right)\sigma_\theta^2. \tag{A.19}$$

In a centralized market

$$V_D\left(m_{n_I},\rho\right)=\frac{1}{2}\frac{1}{\gamma}\frac{n_I\left(n_I+n_D-2\right)}{\left(n_I+n_D-1\right)^2\left(n_I+n_D\right)}\left(n_I+\frac{1-\rho}{\rho}\right)\sigma_\theta^2 \tag{A.20}$$

In a fragmented symmetric market structure, using the expression for welfare in Eq.$(A.19)$ we have

$$\frac{\partial V_D}{\partial\rho}=\left(\frac{n^\ell\beta^\ell}{b^\ell+n^\ell\beta^\ell}\right)^2\left(1+\frac{\gamma}{2}\frac{1}{n_D-1}b^\ell\right)\frac{b^\ell}{n^\ell}\frac{1}{\rho^2}<0,$$

because

$$\begin{aligned}
\left(1+\frac{\gamma}{2}\frac{1}{n_D-1}b^\ell\right) &= 1-\frac{1}{2}\frac{1}{n_D-1}\frac{n^\ell\beta^\ell n_D}{2n^\ell\beta^\ell-n_D} \\
&= \frac{\left(2n_D(n_D-1)+(4-3n_D)n^\ell\beta^\ell\right)}{2(n_D-1)\left(n_D-2n^\ell\beta^\ell\right)}>0,
\end{aligned}$$

since $n_D\geq 3$. In a centralized market structure, using the dealer's welfare in Eq.$(A.20)$ we have

$$\begin{aligned}
\frac{\partial V_D}{\partial\rho} &= \left(\frac{n_I\beta^c}{\beta^c+n_I\beta^c}\right)^2\left(1+\frac{\gamma}{2}\beta^c\right)\frac{\beta^c}{n_I}\frac{1}{\rho^2}\sigma_\theta^2 \\
&= -\frac{1}{2}\frac{1}{\gamma}\frac{n_I-1}{n_I+1}\frac{1}{\rho^2}\frac{1}{n_I}\sigma_\theta^2<0.
\end{aligned}$$

Proof of Proposition 6

Let $\Delta V^D \equiv \mathbb{E}\left(V^{\ell,c}\right) - \mathbb{E}\left(V^{\ell}\right)$. Since β^s is independent of ρ, ΔV^D is a continuous monotone function of ρ. From Lemma A.6 and Lemma A.7 we have

$$\lim_{\rho\to 1} \Delta V^D = \mathbb{E}\left(V^{\ell,c}\right) - \mathbb{E}\left(V^{\ell}\right) > 0 \text{ and } \lim_{\rho\to 0} \Delta V^D = \mathbb{E}\left(V^{\ell,c}\right) - \mathbb{E}\left(V^{\ell}\right) < 0$$

which gives the result.

Lemma A.6

$$\lim_{\rho\to 1} \Delta V^D > 0$$

Proof.

$$\begin{aligned}
\lim_{\rho\to 1} \Delta V^D &= \frac{1}{2\gamma}\left(\frac{n_I^2\left(n_I+n_D-2\right)}{\left(n_I+n_D-1\right)^2\left(n_I+n_D\right)} - \frac{n_S\gamma\beta^s}{\beta^s\gamma n_S-2}\right)\sigma_\theta^2 \\
&= \frac{1}{2\gamma}\left(\frac{n_I^2\left(n_I+n_D-2\right)}{\left(n_I+n_D-1\right)^2\left(n_I+n_D\right)} - \frac{n_S\gamma\beta^s}{\beta^s\gamma n_S-2}\right)\sigma_\theta^2 > 0
\end{aligned}$$

since

$$\frac{n_I^2\left(n_I+n_D-2\right)}{\left(n_I+n_D-1\right)^2\left(n_I+n_D\right)} > \frac{n_S\gamma\beta^s}{\beta^s\gamma n_S-2}.$$

To see this note that

$$\frac{\partial\left(\frac{n_S x}{x n_S-2}\right)}{\partial x} = -2\frac{n_S}{\left(x n_S-2\right)^2} < 0$$

and $-\frac{(n_S-1)}{n_S} > \gamma\beta^s$ since $H\left(-\frac{1}{\gamma}\frac{(n_S-1)}{n_S}\right) = \frac{1}{n_S}\left(n_S-1\right)\left(n_D-2\right) > 0$. Then,

$$-\frac{n_S\gamma\frac{1}{\gamma}\frac{(n_S-1)}{n_S}}{-\frac{1}{\gamma}\frac{(n_S-1)}{n_S}\gamma n_S-2} > \frac{n_S\gamma\beta^s}{\beta^s\gamma n_S-2}$$

and, since,

$$\begin{aligned}
\frac{n_I^2\left(n_I+n_D-2\right)}{\left(n_I+n_D-1\right)^2\left(n_I+n_D\right)} &> -\frac{n_S\gamma\frac{1}{\gamma}\frac{(n_S-1)}{n_S}}{-\frac{1}{\gamma}\frac{(n_S-1)}{n_S}\gamma n_S-2} = \frac{n_S-1}{n_S+1} \\
\left(n_D-1\right)\left(n_S+n_D+n_S n_D-1\right) &> 0
\end{aligned}$$

$\lim_{\rho\to 1} \Delta V^D > 0.$ ■

Lemma A.7 *Dealers are better off in fragmented markets as the correlation in investor valuations disappears, i.e.*

$$\lim_{\rho \to 0} \Delta V^D < 0.$$

Proof.

$$sign\left(\lim_{\rho\to 0}\Delta V^D\right) = sign\left(\frac{n_I\left(n_I+n_D-2\right)}{\left(n_I+n_D-1\right)^2\left(n_I+n_D\right)} - \frac{1}{2}\gamma\beta^s\frac{\left(\left(2\left(n_D-1\right)-\frac{1}{2}n_D\right)n_S\gamma\beta^s - n_D\left(n_D-1\right)\right)}{\left(n_D-\beta^s\gamma n_S\right)^2}\frac{n_D}{n_D-1}\right)$$

Let

$$R\left(\gamma\beta^s, n_D, n_S\right) = \frac{1}{2}\gamma\beta^s\frac{\left(\left(2\left(n_D-1\right)-\frac{1}{2}n_D\right)n_S\gamma\beta^s - n_D\left(n_D-1\right)\right)}{\left(n_D-\beta^s\gamma n_S\right)^2}\frac{n_D}{n_D-1}$$

where

$$\frac{\partial R}{\partial x}\left(x, n_D, n_S\right) = \frac{1}{2}\frac{n_D^2}{\left(n_D-xn_S\right)^3\left(n_D-1\right)}\left(\left(2n_D-3\right)n_Sx - n_D\left(n_D-1\right)\right) < 0 \text{ for all } x<0.$$

Then, since $-\frac{(n_S-1)}{n_S} > \gamma\beta^s$

$$R\left(\gamma\beta^s, n_D, n_S\right) > R\left(-\frac{(n_S-1)}{n_S}, n_D, n_S\right) = \frac{1}{n_S}\left(n_S-1\right)\frac{n_D\left(n_D-1\right)+\left(n_S-1\right)\left(\frac{3}{2}n_D-2\right)}{\left(n_S+n_D-1\right)^2}$$

Also,

$$\frac{\partial R}{\partial n_D}\left(-\frac{(n_S-1)}{n_S}, n_D, n_S\right) = \frac{1}{2}\frac{\left(n_S-1\right)\left(n_S+1\right)}{n_S}\frac{3n_S+n_D-3}{\left(n_S+n_D-1\right)^3} > 0.$$

Then,

$$R\left(-\frac{(n_S-1)}{n_S}, n_D, n_S\right) > R\left(-\frac{(n_S-1)}{n_S}, 3, n_S\right) = \frac{4}{3}\frac{n_S}{n_S+1}\frac{3n_S+1}{\left(3n_S+2\right)^2}.$$

Let

$$R\left(n_D, n_S\right) := \frac{n_S\left(\left(n_S+1\right)n_D-2\right)}{\left(\left(n_S+1\right)n_D-1\right)^2\left(n_S+1\right)}$$

where

$$\frac{\partial L}{\partial n_D}\left(n_D, n_S\right) = -\frac{n_S}{\left(n_D+n_Sn_D-1\right)^3}\left(n_D+n_Sn_D-3\right) < 0.$$

Then,

$$L\left(n_D, n_S\right) < L\left(3, n_S\right) = \frac{4}{3}\frac{n_S}{n_S+1}\frac{3n_S+1}{\left(3n_S+2\right)^2}.$$

Since, as shown in Lemma A.8 below

$$L\left(3,n_S\right)<R\left(-\frac{(n_S-1)}{n_S},3,n_S\right)$$

we have

$$L\left(n_D,n_S\right)<L\left(3,n_S\right)<R\left(-\frac{(n_S-1)}{n_S},3,n_S\right)<R\left(-\frac{(n_S-1)}{n_S},n_D,n_S\right)<R\left(\gamma\beta^\ell,n_D,n_S\right)$$

for all $n_D\geq 3$, which implies

$$\lim_{\rho\to 0}\Delta V^D<0.$$

■

Lemma A.8

$$L\left(3,n_S\right)<R\left(-\frac{(n_S-1)}{n_S},3,n_S\right)$$

Proof. We have

$$L\left(3,n_S\right)-R\left(-\frac{(n_S-1)}{n_S},3,n_S\right)=-\frac{1}{6}\frac{\left(111n_S^3-317\right)n_S^2+\left(265n_S^3-312\right)n_S+\left(49n_S^3-84\right)}{n_S\left(n_S+1\right)\left(n_S+2\right)^2\left(3n_S+2\right)^2}<0,$$

for all $n_S\geq 3$. ■

Proof of Lemma 4

Volume in the inter-dealer market in a fragmented symmetric market structure is

$$\mathcal{V}_D=\frac{1}{2}\mathbb{E}\left[\sum_{\ell=1}^{n_D}\left|q_2^\ell\right|\right]=\frac{1}{2}\mathbb{E}\left[\sum_{\ell=1}^{n_D}\left|\frac{(n_D-2)}{(n_D-1)}\left(\frac{\sum_{l\in N_D}q_1^l}{n_D}-q_1^\ell\right)\right|\right],$$

where

$$q_2^\ell\sim N\left(0,\sigma^2_{q_2^\ell}\right)$$

with

$$\sigma^2_{q_2^\ell}=\frac{\left(n_D-2\right)^2}{\left(n_D-1\right)n_D}\left(\frac{b^\ell\beta^\ell}{n_S\beta^\ell+b^\ell}\right)^2 n_S\frac{1-\rho}{\rho}\sigma^2_\theta,$$

since

$$
\begin{aligned}
q_2^\ell &= \frac{(n_D-2)}{(n_D-1)}\left(\frac{\sum_{l\in N_D} q_1^l}{n_D}-q_1^\ell\right)=\frac{(n_D-2)}{(n_D-1)}b^\ell\left(\frac{\sum_{l\in N_D}\left(p_1^l-p_1^\ell\right)}{n_D}\right)\\
&= \frac{(n_D-2)}{(n_D-1)}\frac{b^\ell\beta^\ell}{n_S\beta^\ell+b^\ell}\left(\frac{\sum_{l\in N_D,l\neq\ell}\left(\sum_{i\in N_I(l)}\eta^i-\sum_{i\in N_I(\ell)}\eta^i\right)}{n_D}\right).
\end{aligned}
$$

$\sigma^2_{q_2^\ell}$ is decreasing in ρ with $\lim_{\rho\to1}\sigma^2_{q_2^\ell}=0$. Since

$$
\mathcal{V}_D=\sqrt{\frac{1}{2\pi}}\sigma_{q_2^\ell} \tag{A.21}
$$

$\frac{\partial\mathcal{V}_D}{\partial\rho}<0$ and $\lim_{\rho\to1}\mathcal{V}_D=0$.

Proof of Lemma 5

The volume traded in the local markets in a fragmented market structure is $n_D\mathcal{V}^\ell$ where

$$
\mathcal{V}^\ell=\frac{1}{2}\mathbb{E}\left[\frac{\sum_{i\in N_I(\ell)}\left|x_{1i}^\ell\right|+\left|q_1^\ell\right|}{n^\ell+1}\right].
$$

We know that

$$
x_1^\ell\sim N\left(0,\sigma^2_{x_1^\ell}\right)\text{ and }q_1^\ell\sim N\left(0,\sigma^2_{q_1^\ell}\right),
$$

where

$$
\begin{aligned}
\sigma^2_{x_1} &= \mathbb{V}ar\left(x_1^\ell\right)=\mathbb{V}ar\left(-\beta^\ell\left(\theta^i-p_1^\ell\right)\right)\\
&= \beta^{\ell2}\left(\mathbb{V}ar\left(\theta^i\right)+\mathbb{V}ar\left(p_1^\ell\right)+2\mathbb{C}ov\left(\theta^i,p_1^\ell\right)\right)\\
&= \frac{\beta^{\ell2}}{\left(b^\ell+n^\ell\beta^\ell\right)^2}\left(\left(a^\ell+b^\ell\right)^2+\left(\left(\left(n^\ell-1\right)\beta^\ell+b^\ell\right)^2+\left(n^\ell-1\right)\beta^{\ell2}\right)\frac{1-\rho}{\rho}\right)\sigma^2_\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma^2_{q_1} &= \mathbb{V}ar\left(q_1^\ell\right)=\mathbb{V}ar\left(a^\ell\theta+b^\ell p_1^\ell\right)\\
&= a^{\ell2}\mathbb{V}ar\left(\theta\right)+b^{\ell2}\mathbb{V}ar\left(p_1^\ell\right)+2a^\ell b^\ell\mathbb{C}ov\left(\theta,p_1^\ell\right)\\
&= \frac{n^\ell\beta^{\ell2}}{\left(b^\ell+n^\ell\beta^\ell\right)^2}\left(n^\ell\left(a^\ell+b^\ell\right)^2+b^{\ell2}\frac{1-\rho}{\rho}\right)\sigma^2_\theta.
\end{aligned}
$$

Then, since for $x \sim N(0, \sigma^2)$, $|x|$ is a folded normal with $\mathbb{E}[|x|] = \sqrt{\frac{2}{\pi}}\, \sigma$, we have

$$\mathcal{V}^{\ell} = \frac{1}{\sqrt{2\pi}} \left(\frac{n^{\ell} \sigma_{x_1} + \sigma_{q_1^{\ell}}}{n^{\ell} + 1} \right). \tag{A.22}$$

In a centralized market, volume is given by

$$\mathcal{V}_c = \frac{1}{2} \mathbb{E} \left[\frac{\sum_{i=1}^{n_I} |x_{1i}^c| + n_D |q_1^c|}{n_I + n_D} \right].$$

We know that

$$x_{1i}^c \sim N\left(0, \sigma_{x_1^c}^2\right) \text{ and } q_1^c \sim N\left(0, \sigma_{q_1^c}^2\right),$$

where

$$\begin{aligned}
\sigma_{x_1^c}^2 &= \mathbb{V}ar\left(x_1^c\right) = \mathbb{V}ar\left(-\beta^c\left(\theta^i - p_1^c\right)\right) \\
&= \beta^{c2}\left(\mathbb{V}ar\left(\theta^i\right) + \mathbb{V}ar\left(p_1^{\ell}\right) - 2\mathbb{C}ov\left(\theta^i, p_1^{\ell}\right)\right) \\
&= \beta^{c2} \frac{1}{(n_I + 1)^2} \left(1 + \left(n_I^2 + (n_I - 1)\right) \frac{1 - \rho}{\rho}\right) \sigma_\theta^2
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{q_1^c}^2 &= \mathbb{V}ar\left(q_1^{\ell}\right) = \mathbb{V}ar\left(\beta^c p_1^{\ell}\right) \\
&= \beta^{c2} \mathbb{V}ar\left(p_1^{\ell}\right) = \beta^{c2} \frac{1}{(n_I + 1)^2} \left(n_I^2 + n_I \frac{1 - \rho}{\rho}\right) \sigma_\theta^2.
\end{aligned}$$

Then,

$$\mathcal{V}_c = \frac{1}{\sqrt{2\pi}} \left(\frac{n_I \sigma_{x_1^c} + n_D \sigma_{q_1^c}}{n_I + n_D} \right) \tag{A.23}$$

Because a^{ℓ}, β^{ℓ}, and b^{ℓ} do not depend on ρ, a) and b) follow directly from Eq.$A.22$ and Eq.$A.21$, respectively since $\sigma_{x_1^i}$, $\sigma_{q_1^{\ell}}$, and $\sigma_{q_2^{\ell}}$ are decreasing in ρ. The third claim c) follows from Eq.$A.23$ since $\sigma_{x_1^c}$ and $\sigma_{q_1^c}$ are decreasing in ρ because β^c does not depend on ρ.

B Appendix: Supplementary Derivations and Results

Lemma B.9 ***(Characterization of*** β^{ℓ} ***and of*** υ_1^{ℓ}***)*** *Investors' price sensitivity* $\beta^{\ell} \in \left(-\frac{1}{\gamma}, 0\right)$ *satisfies:*

a) $\frac{\partial \beta^{\ell}}{\partial n^{\ell}} < 0$;

b) $\frac{\partial \beta^{\ell}}{\partial n_D} < 0$;

c) $\bar{\beta}^{\ell} \equiv \lim_{n_D \to \infty} \beta^{\ell} \equiv -\frac{2\left(n^{\ell}-1\right)}{\gamma\left(2n^{\ell}-1\right)}$ *and* $\lim_{n \to \infty} \beta^{\ell} = -\frac{1}{\gamma}$.

In addition,

$$\frac{\partial \upsilon_1^{\ell}}{\partial n^{\ell}} < 0.$$

Proof. Recall that $\beta\left(n\right)$ is defined by the negative root of

$$H\left(\beta\right) = -2n\left(n-1\right)\left(\gamma\beta\right)^2 + \left(\left(2n-1\right)n_D - 2n\left(n-2\right)\right)\gamma\beta + 2\left(n-1\right)n_D.$$

Since $\beta\left(n\right)$ is the negative root of H and H is concave and $H\left(0\right) > 0$, $H'\left(\beta\left(n\right)\right) > 0$. Also,

$$H\left(-\frac{1}{\gamma}\right) = -2n\left(n-1\right) - \left(\left(2n-1\right)n_D - 2n\left(n-2\right)\right) + 2\left(n-1\right)n_D = -2n - n_D < 0,$$

which implies that $\beta\left(n\right) > -\frac{1}{\gamma}$.

a) Using the implicit function theorem

$$\frac{\partial \beta\left(n\right)}{\partial n} = -\frac{\frac{\partial H(\beta)}{\partial n}|_{\beta(n)}}{H'\left(\beta\left(n\right)\right)} = -\frac{2n_D + 2\left(1-2n\right)\left(\gamma\beta\left(n\right)\right)^2 + 2\left(n_D - 2\left(n-1\right)\right)\beta\left(n\right)\gamma}{-\gamma\left(-\left(2n-1\right)n_D + 2n\left(n-2\right) + 4n\left(n-1\right)\beta\left(n\right)\gamma\right)}$$

and

$$sign\left(\frac{\partial \beta\left(n\right)}{\partial n}\right) = -sign\left(n_D - \left(2n-1\right)\left(\gamma\beta\left(n\right)\right)^2 + \left(n_D - 2\left(n-1\right)\right)\beta\left(n\right)\gamma\right).$$

Let

$$X \equiv n_D - \left(2n-1\right)\left(\gamma\beta\left(n\right)\right)^2 + \left(n_D - 2\left(n-1\right)\right)\beta\left(n\right)\gamma$$

From the definition of $\beta(n)$ we have

$$(\gamma\beta(n))^2 = \frac{((2n-1)n_D - 2n(n-2))\gamma\beta(n) + 2(n-1)n_D}{2n(n-1)}.$$

Then,

$$X = -\frac{1}{2n(n-1)}\left(\left(n_D(n-1)^2 + (n_D+2)n^2\right)\gamma\beta(n) + 2n_D(n-1)^2\right)$$

Moreover,

$$H\left(\frac{-2n_D(n-1)^2}{\gamma\left(n_D(n-1)^2 + (n_D+2)n^2\right)}\right) = \frac{2nn_D(n-1)\left(\left(2n^2-2n+1\right)n_D^2+4n\left(n^2-n+1\right)n_D+4n^2\left(n^2-2n+2\right)\right)}{\left(n_D(n-1)^2+(n_D+2)n^2\right)^2} > 0$$

since $(2n^2-2n+1)n_D^2 + 4n(n^2-n+1)n_D + 4n^2(n^2-2n+2)$ admits complex roots in n_D for all $n \geq 1$ and it is positive at $n_D = 0$.

Then,

$$\gamma\beta(n) < \frac{-2n_D(n-1)^2}{\gamma\left(n_D(n-1)^2 + (n_D+2)n^2\right)} \text{ and } X > 0,$$

which implies $\frac{\partial\beta(n)}{\partial n} < 0$.

b) Also,

$$\frac{\partial\beta(n)}{\partial n_D} = -\frac{\frac{\partial H(\beta)}{\partial n_D}|_{\beta(n)}}{H'(\beta(n))} = -\frac{(2n-1)\gamma\beta + 2(n-1)}{-\gamma(-(2n-1)n_D + 2n(n-2) + 4n(n-1)\beta(n)\gamma)}$$

and

$$sign\left(\frac{\partial\beta(n)}{\partial n_D}\right) = -sign\left((2n-1)\gamma\beta(n) + 2(n-1)\right).$$

Since

$$H\left(-\frac{2(n-1)}{(2n-1)}\right) = -4\frac{n^2}{(2n-1)^2}(n-1) < 0.$$

we have $\gamma\beta(n) > -\frac{2(n-1)}{(2n-1)}$ and, therefore,

$$sign\left(\frac{\partial\beta(n)}{\partial n_D}\right) = -sign\left((2n-1)\gamma\beta(n) + 2(n-1)\right) < 0.$$

c) Finally, from the definition of $\beta(n)$ we have

$$\beta(n) = -\frac{1}{\gamma}\frac{1-\sqrt{1+\frac{16n(n-1)^2 n_D}{((2n-1)n_D-2n(n-2))^2}}}{-\frac{4n(n-1)}{((2n-1)n_D-2n(n-2))}} = -\frac{1}{\gamma}\frac{\frac{4(n-1)n_D}{((2n-1)n_D-2n(n-2))}}{\left(1+\sqrt{1+\frac{16n(n-1)^2 n_D}{((2n-1)n_D-2n(n-2))^2}}\right)}.$$

Then, taking limits gives

$$\lim_{n_D\to\infty}\beta(n)=-\frac{2(n-1)}{\gamma(2n-1)}\equiv\bar{\beta}\text{ and }\lim_{n\to\infty}\beta(n)=-\frac{1}{\gamma}.$$

Finally, recall that

$$\upsilon_1^\ell=-\frac{1}{(n^\ell-1)\beta^\ell+b^\ell}.$$

Thus

$$\frac{\partial\upsilon_1^\ell}{\partial n^\ell}=\left(\frac{1}{(n^\ell-1)\beta^\ell+b^\ell}\right)^2\left(\beta^\ell+\left(n^\ell-1\right)\frac{\partial\beta^\ell}{\partial n^\ell}+\frac{\partial b^\ell}{\partial n^\ell}\right)$$

From Lemma $B.10$ bellow we have that $\frac{\partial b^\ell}{\partial n^\ell}<0$, which implies that

$$\frac{\partial\upsilon_1^\ell}{\partial n^\ell}<0.$$

■

Lemma B.10 ***(Characterization of*** b^ℓ ***and of*** λ_1^ℓ***)*** *Dealers' price sensitivity* b^ℓ *satisfies*

a) $\frac{\partial b^\ell}{\partial n^\ell}<0$*;*

b) $\frac{\partial b^\ell}{\partial n_D}<0$*;*

c) $\lim_{n_D\to\infty}b^\ell=-\frac{2n^\ell\left(n^\ell-1\right)}{\gamma\left(2n^\ell-1\right)}$ *and* $\lim_{n^\ell\to\infty}b^\ell=-\frac{n_D}{2\gamma}$.

In addition

$$\frac{\partial\lambda_1^\ell}{\partial n^\ell}<0.$$

Proof. Using the definition of b^ℓ in Eq.($A.4$) and Lemma $B.9$ we have

$$\text{a) }\frac{\partial b^\ell}{\partial n^\ell}=\frac{n_D^2}{\left(2\gamma n^\ell\beta^\ell-n_D\right)^2}\left(n^\ell\frac{\partial\beta^\ell}{\partial n^\ell}+\beta^\ell\right)<0,$$

and

$$\text{b) }\frac{\partial b^\ell}{\partial n_D}=\frac{-2\left(n^\ell\beta^\ell\right)^2\gamma+n^\ell n_D^2\frac{\partial\beta^\ell}{\partial n_D}}{\left(n_D-2n^\ell\beta^\ell\gamma\right)^2}<0.$$

c). Moreover,

$$\lim_{n_D\to\infty}b^\ell=\lim_{n_D\to\infty}-\frac{n^\ell\beta^\ell}{\frac{2\gamma n^\ell\beta^\ell}{n_D}-1}=n^\ell\bar{\beta}^\ell=-\frac{2n^\ell\left(n^\ell-1\right)}{\gamma\left(2n^\ell-1\right)}$$

and

$$\lim_{n^\ell \to \infty} b^\ell = \lim_{n^\ell \to \infty} -\frac{\beta^\ell n_D}{2\gamma\beta^\ell - \frac{n_D}{n^\ell}} = -\frac{n_D}{2\gamma}.$$

Finally, recall that

$$\lambda_1^\ell = -\frac{1}{n^\ell \beta^\ell}$$

Thus

$$\frac{\partial \lambda_1^\ell}{\partial n^\ell} = \left(\frac{1}{n^\ell \beta^\ell}\right)^2 \left(\beta^\ell + \frac{\partial \beta^\ell}{\partial n^\ell} n^\ell\right) < 0.$$

■

Lemma B.11 ***(Price sensitivities)*** *$\beta^c < \beta^\ell$.*

Proof. Note that

$$\beta^c = -\frac{1}{\gamma}\frac{(n_I + n_D - 2)}{(n_I + n_D - 1)} < -\frac{1}{\gamma}\frac{(n_I - 1)}{n_I}.$$

Since β^ℓ is given by $H\left(\beta^\ell\right) = 0$ where

$$H(\beta) = -2n(n-1)(\gamma\beta)^2 + ((2n-1)n_D - 2n(n-2))\gamma\beta + 2(n-1)n_D.$$

and

$$H\left(-\frac{1}{\gamma}\frac{(n_I - 1)}{n_I}\right) = -\frac{1}{n_S n_D^2}\left(2n_S^2 n_D (n_D - 1) + n_S n_D^2 (n_D - 2) + 2(n_S - 1) + n_D^2\right) < 0,$$

we have $\beta^s > -\frac{1}{\gamma}\frac{(n_I - 1)}{n_I} > \beta^c$ since $\beta^c < 0$, $H'' < 0$ and $H(0) > 0$. ■

Lemma B.12 ***(Noise in price)*** *$\pi_\eta^s > \pi_\eta^c$*

Proof.

$$\begin{aligned}
\pi_\eta^s &> \pi_\eta^c \\
\frac{1}{n_S}\frac{n_D - 2\beta^s\gamma n_S}{2n_D - 2\beta^s\gamma n_S} &> \frac{1}{(n_S+1)n_D} \\
(n_S+1)n_D(n_D - 2\beta^s\gamma n_S) &> 2n_S(n_D - \beta^s\gamma n_S) \\
(n_S+1)n_D n_D - 2(n_S+1)n_D\beta^s\gamma n_S &> 2n_S n_D - 2n_S\beta^s\gamma n_S \\
-2((n_D-1)n_S + n_D)\beta^s\gamma n_S &> -\left(n_D(n_D-2)n_S + n_D^2\right) \\
\beta^s\gamma n_S &< \frac{(n_D(n_D-2)n_S + n_D^2)}{2((n_D-1)n_S + n_D)}
\end{aligned}$$

which always holds since $\beta^s < 0$ and the right hand side of this expression is positive. ■

C Online appendix (not for publication)

This section contains intermediate results used in the Appendix.

Lemma C.13 *In a symmetric equilibrium*

$$a^{sym} = -n^{\ell 2} \beta^{sym2} \frac{\gamma}{n^{\ell} \beta^{sym} \gamma - 2} \frac{n_D - 2}{n_D - 2n^{\ell} \beta^{sym} \gamma} \tag{C.1}$$

where

$$\lim_{n_D \to \infty} a^{sym} = \frac{\gamma \left(n^{\ell} \bar{\beta}\right)^2}{2 - \gamma \bar{\beta} n^{\ell}} \ , \quad \lim_{n^{\ell} \to \infty} a^{sym} = \lim_{n^{\ell} \to \infty} -b^{sym} = \frac{n_D}{2\gamma}.$$

$$\lim_{n_D \to 3} a^{sym} = \frac{\frac{1}{3} n^{\ell} \left(b^{sym}\right)^2 \gamma \beta^{sym}}{\left(n^{\ell} \beta^{sym} + b^{sym} - \frac{1}{3} b^{sym} n^{\ell}\right)}.$$

If one investor deviates from a fragmented symmetric market structure, then

$$\lim_{n_D \to \infty} \begin{bmatrix} a^{\ell} \\ a^{h} \\ a^{o} \end{bmatrix} = \begin{bmatrix} -\gamma \frac{n^{\ell} \beta^{o}}{n^{\ell} \beta^{o} \gamma - 2} \left(n^{\ell} - 1\right) \bar{\beta}^{\ell} \\ -\gamma \frac{n^{\ell} \beta^{o}}{n^{\ell} \beta^{o} \gamma - 2} \left(n^{\ell} + 1\right) \bar{\beta}^{h} \\ -\gamma \frac{n^{\ell} \beta^{o}}{n^{\ell} \beta^{o} \gamma - 2} n^{\ell} \beta^{o} \end{bmatrix}$$

and

$$\lim_{n_D \to 3} \begin{bmatrix} a^{\ell} \\ a^{h} \\ a^{o} \end{bmatrix} = \begin{bmatrix} b^{\ell} \gamma \left(5 b^{\ell} \gamma + 12\right) \frac{5 b^{h+o} \gamma + 6 b^{h} + 6 b^{o}}{90 b^{h+o} \gamma^2 + 288 b^{h} \gamma + 288 b^{o} \gamma + 90 b^{h+\ell} \gamma^2 + 90 b^{o+\ell} \gamma^2 + 25 b^{h+o+\ell} \gamma^3 + 288 b^{\ell} \gamma + 864} \\ b^{h} \gamma \left(5 b^{h} \gamma + 12\right) \frac{6 b^{\ell} + 5 b^{o+\ell} \gamma + 6 b^{o}}{90 b^{h+o} \gamma^2 + 288 b^{h} \gamma + 288 b^{o} \gamma + 90 b^{h+\ell} \gamma^2 + 90 b^{o+\ell} \gamma^2 + 25 b^{h+o+\ell} \gamma^3 + 288 b^{\ell} \gamma + 864} \\ b^{o} \gamma \left(5 b^{o} \gamma + 12\right) \frac{6 b^{\ell} + 5 b^{h+\ell} \gamma + 6 b^{h}}{90 b^{h+o} \gamma^2 + 288 b^{h} \gamma + 288 b^{o} \gamma + 90 b^{h+\ell} \gamma^2 + 90 b^{o+\ell} \gamma^2 + 25 b^{h+o+\ell} \gamma^3 + 288 b^{\ell} \gamma + 864} \end{bmatrix}$$

Proof. In a symmetric equilibrium

$$a^{sym} = \frac{(n_D - 2)}{n_D} b^{sym} n^{\ell} \frac{a^{sym} + b^{sym}}{n^{\ell} \beta^{sym} + b^{sym}} \gamma \beta^{sym}$$

or, alternatively,

$$a^{sym} = \frac{\frac{(n_D - 2)}{n_D} n^{\ell} \left(b^{sym}\right)^2 \gamma \beta^{sym}}{\left(n^{\ell} \beta^{sym} + b^{sym} - \frac{(n_D - 2)}{n_D} b^{sym} n^{\ell} \gamma \beta^{sym}\right)}$$

where

$$b^{sym} = -\frac{n^{\ell} \beta^{sym} n_D}{2\gamma n^{\ell} \beta^{sym} - n_D}$$

and $\beta^{sym} = \beta\left(n^{\ell}\right)$. Then,

$$\lim_{n_D\to\infty} a^{sym} = \lim_{n_D\to\infty} \frac{\frac{(n_D-2)}{n_D} n^{\ell}\left(b^{sym}\right)^2 \gamma\beta^{sym}}{\left(n^{\ell}\beta^{sym} + b^{sym} - \frac{(n_D-2)}{n_D} b^{sym} n^{\ell}\gamma\beta^{sym}\right)} = \frac{\gamma\left(n^{\ell}\bar{\beta}\right)^2}{2-\gamma\bar{\beta}n^{\ell}}$$

$$= \frac{\gamma\left(n^{\ell}\frac{2\left(n^{\ell}-1\right)}{\gamma\left(2n^{\ell}-1\right)}\right)^2}{2+\gamma\frac{2\left(n^{\ell}-1\right)}{\gamma\left(2n^{\ell}-1\right)}n^{\ell}} = \frac{2n^{2\ell}\left(n^{\ell}-1\right)^2}{\gamma\left(2n^{\ell}-1\right)\left(n^{\ell}+n^{2\ell}-1\right)}$$

$$\lim_{n^{\ell}\to\infty} a^{sym} = \lim_{n^{\ell}\to\infty} -b^{sym} = \frac{n_D}{2\gamma}$$

$$\lim_{n^{\ell}\to 3} a^{sym} = \frac{\frac{1}{3}\left(b^{sym}\right)^2 \gamma n^{\ell}\beta^{sym}}{\left(n^{\ell}\beta^{sym} + b^{sym} - \frac{1}{3}b^{sym}n^{\ell}\right)}$$

Using that

$$n^{\ell}\beta^{sym} = \frac{3b^{sym}}{\left(2\gamma b^{sym}+3\right)}$$

$$\lim_{n^{\ell}\to 3} a^{sym} = \frac{\left(b^{sym}\right)^2\gamma}{\left(3+\left(1-\frac{1}{3}n^{\ell}\right)\left(2\gamma b^{sym}+3\right)\right)}$$

$$= \frac{\left(b^{sym}\right)^2\gamma}{\left(6+\left(1-\frac{1}{3}n^{\ell}\right)2\gamma b^{sym}-n^{\ell}\right)}$$

If the investor chooses to deviate, the market structure is as follows: market ℓ has $n^{\ell}-1$ investors, market h has $n^{\ell}+1$ and the rest of the n_D-2 markets have n^{ℓ} investors. Then,

$$a^{\ell} = b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\sum_{l\in N_D, l\neq\ell} n^{l}\frac{a^{l}\beta^{l}-b^{l}\alpha^{l}}{n^{l}\beta^{l}+b^{l}}$$

$$a^{\ell} = b^{\ell}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-2)\,n^{\ell}\frac{a^{o}+b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o} + \left(n^{\ell}+1\right)\frac{a^{h}+b^{h}}{\left(n^{\ell}+1\right)\beta^{h}+b^{h}}\beta^{h}\right)$$

$$a^{h} = b^{h}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-2)\,n^{\ell}\frac{a^{o}+b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o} + \left(n^{\ell}-1\right)\frac{a^{\ell}+b^{\ell}}{\left(n^{\ell}-1\right)\beta^{\ell}+b^{\ell}}\beta^{\ell}\right)$$

$$a^{o} = b^{o}\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-3)\,n^{\ell}\frac{a^{o}+b^{o}}{n^{\ell}\beta^{o}+b^{o}}\beta^{o} + \left(n^{\ell}-1\right)\frac{a^{\ell}+b^{\ell}}{\left(n^{\ell}-1\right)\beta^{\ell}+b^{\ell}}\beta^{\ell} + \left(n^{\ell}+1\right)\frac{a^{h}+b^{h}}{\left(n^{\ell}+1\right)\beta^{h}+b^{h}}\beta^{h}\right)$$

One can rewrite this system as

$$\begin{bmatrix} a^{\ell} \\ a^{h} \\ a^{o} \end{bmatrix} = \left[I_3 - \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}\right]^{-1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{C.2}$$

where

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} = \begin{bmatrix} 0 & b^\ell\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{\left(n^\ell+1\right)\beta^h}{\left(n^\ell+1\right)\beta^h+b^h} & b^\ell\gamma\frac{(n_D-2)}{n_D}\frac{(n_D-2)}{n_D-1}\frac{n^\ell\beta^o}{n^\ell\beta^o+b^o} \\ b^h\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{\left(n^\ell-1\right)\beta^\ell}{\left(n^\ell-1\right)\beta^\ell+b^\ell} & 0 & b^h\gamma\frac{(n_D-2)}{n_D}\frac{(n_D-2)}{n_D-1}\frac{n^\ell\beta^o}{n^\ell\beta^o+b^o} \\ b^o\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{\left(n^\ell-1\right)\beta^\ell}{\left(n^\ell-1\right)\beta^\ell+b^\ell} & b^o\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\frac{\left(n^\ell+1\right)\beta^h}{\left(n^\ell+1\right)\beta^h+b^h} & b^o\gamma\frac{(n_D-2)}{n_D}\frac{(n_D-3)}{n_D-1}\frac{n^\ell\beta^o}{n^\ell\beta^o+b^o} \end{bmatrix}$$

and

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b^\ell\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-2)\frac{n^\ell b^o}{n^\ell\beta^o+b^o}\beta^o + \frac{\left(n^\ell+1\right)b^h}{\left(n^\ell+1\right)\beta^h+b^h}\beta^h\right) \\ b^h\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-2)\frac{n^\ell b^o}{n^\ell\beta^o+b^o}\beta^o + \frac{\left(n^\ell-1\right)b^\ell}{\left(n^\ell-1\right)\beta^\ell+b^\ell}\beta^\ell\right) \\ b^o\gamma\frac{(n_D-2)}{n_D}\frac{1}{n_D-1}\left((n_D-3)\frac{n^\ell b^o}{n^\ell\beta^o+b^o}\beta^o + \frac{\left(n^\ell-1\right)b^\ell}{\left(n^\ell-1\right)\beta^\ell+b^\ell}\beta^\ell + \frac{\left(n^\ell+1\right)b^h}{\left(n^\ell+1\right)\beta^h+b^h}\beta^h\right) \end{bmatrix}.$$

Then,

$$\lim_{n_D\to\infty}\begin{bmatrix} a^\ell \\ a^h \\ a^o \end{bmatrix} = \left[I_3 - \lim_{n_D\to\infty}\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}\right]^{-1}\lim_{n_D\to\infty}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -n^\ell\beta^{o+\ell}\gamma\frac{n^\ell-1}{n^\ell\beta^o\gamma-2} \\ -n^\ell\beta^{h+o}\gamma\frac{n^\ell+1}{n^\ell\beta^o\gamma-2} \\ -n^{2\ell}\beta^{2o}\frac{\gamma}{n^\ell\beta^o\gamma-2} \end{bmatrix}$$

$$\lim_{n_D\to\infty}\begin{bmatrix} a^\ell \\ a^h \\ a^o \end{bmatrix} = \left[I_3 - \lim_{n_D\to\infty}\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}\right]^{-1}\lim_{n_D\to\infty}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\gamma}\frac{\left(n^\ell-1\right)^2\left(n^\ell-2\right)}{\left(2n^\ell-3\right)\left(n^{\ell 2}+n^\ell-1\right)} \\ \frac{2}{\gamma}\frac{n^\ell\left(n^\ell-1\right)\left(n^\ell+1\right)}{\left(2n^\ell+1\right)\left(n^{\ell 2}+n^\ell-1\right)} \\ \frac{2}{\gamma}\frac{n^\ell\left(n^\ell-1\right)^2}{\left(2n^\ell-1\right)\left(n^{\ell 2}+n^\ell-1\right)} \end{bmatrix}$$

When $n_D \to \infty$, we have

$$\begin{aligned}\lim_{n_D\to 3}\begin{bmatrix} a^\ell \\ a^h \\ a^o \end{bmatrix} &= \left[I_3 - \lim_{n_D\to\infty}\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}\right]^{-1}\lim_{n_D\to\infty}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} b^\ell\gamma\left(5b^\ell\gamma+12\right)\frac{5b^{h+o}\gamma+6b^h+6b^o}{90b^{h+o}\gamma^2+288b^h\gamma+288b^o\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^\ell\gamma+864} \\ b^h\gamma\left(5b^h\gamma+12\right)\frac{6b^\ell+5b^{o+\ell}\gamma+6b^o}{90b^{h+o}\gamma^2+288b^h\gamma+288b^o\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^\ell\gamma+864} \\ b^o\gamma\left(5b^o\gamma+12\right)\frac{6b^\ell+5b^{h+\ell}\gamma+6b^h}{90b^{h+o}\gamma^2+288b^h\gamma+288b^o\gamma+90b^{h+\ell}\gamma^2+90b^{o+\ell}\gamma^2+25b^{h+o+\ell}\gamma^3+288b^\ell\gamma+864} \end{bmatrix}\end{aligned}$$

where we used that

$$n\beta(n) = \frac{3b(n)}{(2\gamma b(n) + 3)}.$$

■

Lemma C.14 *Let*

$$F(n) \equiv -\frac{(4\gamma b(n)\,n + 6n + 3\gamma b(n))}{n(\gamma b(n) + 6)^2} b(n).$$

Then, $F(n) > 0$ for all $n \geq 0$ and $F'(n) < 0$.

Proof. First, note that

$$(4n + 3)\gamma b(n) + 6n > 0 \tag{C.3}$$

since using that $b(n) = \frac{3n\beta(n)}{(3-2n\gamma\beta(n))}$ Eq.$(C.3)$ becomes

$$(4n + 3)\gamma 3n\beta(n) + 6n(3 - 2n\gamma\beta(n)) = 9n(\gamma\beta(n) + 2) > 0$$

since $\beta(n) \geq -\frac{1}{\gamma}$. The derivative of $F(\cdot)$ with respect to n is

$$F'(n) = \frac{3}{n(b(n)\gamma + 6)^2}\left(\gamma\frac{b(n)^2}{n} - \frac{2}{(b(n)\gamma + 6)}(6n + 6b(n)\gamma + 7b(n)n\gamma)\frac{\partial b(n)}{\partial n}\right)$$

Then, using that $b(n) = \frac{3n\beta(n)}{(3-2n\gamma\beta(n))}$ we have

$$\frac{\partial b(n)}{\partial n} = \frac{9}{(2n\gamma\beta(n) - 3)^2}\left(\beta(n) + 9n\frac{\partial\beta(n)}{\partial n}\right)$$

and

$$F'(n) = \frac{n}{3n^2(n\gamma\beta(n) - 2)^3}\left((\beta(n)\gamma + 2)(n\beta(n)\gamma + 2)\beta(n) + 18n((n+2)\gamma\beta(n) + 2)\frac{\partial\beta(n)}{\partial n}\right).$$

From Lemma $B.9$ we know $\beta(n) \geq -\frac{1}{\gamma}$ and thus $(\beta(n)\gamma + 2) > 0$. Moreover, $n\beta(n)\gamma + 2 < 0$ since

$$H\left(-\frac{2}{n\gamma}; n_D = 3\right) = \frac{2}{n}\left(5n^2 - 17n + 7\right) > 0 \text{ for } n \geq 3.$$

Then, $(n + 2)\gamma\beta(n) + 2 < 0$ and $F'(n) < 0$. ■

Lemma C.15 *We show that*

$$\frac{n^{\ell}+1}{n^{\ell}} > \left(\frac{\left(\gamma b^{dev}+6\right)}{\left(2\gamma b^{dev}+6\right)}\left(1+\frac{a^{dev}}{b^{dev}}\right)\right)^2. \tag{C.4}$$

Proof. Using Lemma $C.13$ we have

$$\frac{\left(\gamma b^{dev}+6\right)}{\left(2\gamma b^{dev}+6\right)}\left(1+\frac{a^{dev}}{b^{dev}}\right) = \frac{\left(\gamma b^{dev}+6\right)\left(5\gamma b^{sym}+12\right)\left(5\gamma b^{\ell}+12\right)}{288\gamma b^{sym}+288\gamma b^{dev}+288\gamma b^{\ell}+90\gamma^2 b^{sym}b^{dev}+90\gamma^2 b^{sym}b^{\ell}+90\gamma^2 b^{dev}b^{\ell}+25\gamma^3 b^{sym}b^{dev}b^{\ell}+864}.$$

Then, showing Eq.$(C.4)$ holds is the same as showing that

$$\begin{aligned}
\left(\frac{288\gamma b^{sym}+288\gamma b^{dev}+288\gamma b^{\ell}+90\gamma^2 b^{sym}b^{dev}+90\gamma^2 b^{sym}b^{\ell}+90\gamma^2 b^{dev}b^{\ell}+25\gamma^3 b^{sym}b^{dev}b^{\ell}+864}{\left(\gamma b^{dev}+6\right)\left(5\gamma b^{sym}+12\right)\left(5\gamma b^{\ell}+12\right)}\right)^2 &> \frac{n^{\ell}}{n^{\ell}+1} \\
\left(-6\gamma\frac{\left(b^{\ell}-b^{dev}\right)}{\left(\gamma b^{dev}+6\right)\left(5\gamma b^{\ell}+12\right)}-6\gamma\frac{\left(b^{sym}-b^{dev}\right)}{\left(5b^{sym}\gamma+12\right)\left(\gamma b^{dev}+6\right)}+1\right)^2 &> \frac{n^{\ell}}{n^{\ell}+1} \\
\left(-G\left(b^{\ell};b^{dev}\right)-G\left(b^{sym};b^{dev}\right)+1\right)^2 &> \frac{n^{\ell}}{n^{\ell}+1},
\end{aligned} \tag{C.5}$$

where

$$G\left(x;b^{dev}\right)=6\gamma\frac{\left(x-b^{dev}\right)}{\left(\gamma b^{dev}+6\right)\left(5\gamma x+12\right)}.$$

Because

$$b\left(n\right)=\frac{3n\beta\left(n\right)}{\left(3-2\gamma n\beta\left(n\right)\right)}, \tag{C.6}$$

we have

$$\frac{\left(5\gamma b\left(n\right)+12\right)}{\left(\gamma b\left(n\right)+6\right)}=\frac{\beta\left(n\right)\gamma\left(n+1\right)-4}{\beta\left(n\right)\gamma\left(n+1\right)-2}>0,$$

which implies

$$G'\left(x\right)=\frac{\left(5\gamma b^{dev}+12\right)}{\left(\gamma b^{dev}+6\right)\left(5x\gamma+12\right)^2}>0$$

and $G\left(x,b^{dev}\right)>0$ for $x>b^{dev}$ since $G\left(b^{dev};b^{dev}\right)=0$. Then, since $G\left(b^{\ell};b^{dev}\right)>G\left(b^{sym};b^{dev}\right)$, if

$$2G\left(b^{\ell};b^{dev}\right)=12\gamma\frac{\left(b^{\ell}-b^{dev}\right)}{\left(5b^{\ell}\gamma+12\right)\left(\gamma b^{dev}+6\right)}<1-\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}, \tag{C.7}$$

Eq.$(C.5)$ holds. Using that $\gamma\beta^{dev}\in(-1,0)$ and $(C.6)$ we have

$$b\left(n\right)>-\frac{1}{\gamma}\frac{3n}{\left(3+2n\right)}.$$

Because the left hand side of Eq.$(C.7)$ is decreasing in b^{dev} we can rewrite Eq.$(C.7)$ as

$$
\begin{aligned}
4\frac{\left((2n+5)\,\gamma b^{\ell}+3\,(n+1)\right)}{3\,(5b^{\ell}\gamma+12)\,(n+3)} &< 1-\sqrt{\frac{n^{\ell}}{n^{\ell}+1}} \\
\gamma b^{\ell} &< -\frac{12\left(n^{\ell}+1+3\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)\right)}{\left(4\left(2n^{\ell}+5\right)+15\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)\right)}
\end{aligned}
$$

since $4\left(2n^{\ell}+5\right)+15\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)>0$. Using Eq.$(C.6)$ this becomes

$$
\begin{aligned}
\frac{3\left(n^{\ell}-1\right)\gamma\beta^{\ell}}{\left(3-2\gamma\left(n^{\ell}-1\right)\beta^{\ell}\right)} &< -\frac{12\left(n^{\ell}+1+3\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)\right)}{\left(4\left(2n^{\ell}+5\right)+15\left(\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}-1\right)\left(n^{\ell}+3\right)\right)} \\
\gamma\beta^{\ell} &< \frac{4\left(-2\left(n^{\ell}+4\right)+3\left(3+n^{\ell}\right)\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}\right)}{\left(n^{\ell}-1\right)\left(-3n^{\ell}-13+3\left(3+n^{\ell}\right)\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}\right)} \equiv Z
\end{aligned}
$$

which holds because

$$
H\left(\frac{Z}{\gamma};n=n^{\ell}-1,n_D=3\right)=-2\left(n^{\ell}-1\right)\left(n^{\ell}-2\right)Z^2+\left(3\left(2n^{\ell}-3\right)+2\left(n^{\ell}-1\right)-2\left(n^{\ell}-1\right)\left(n^{\ell}-2\right)\right)Z+6\left(n^{\ell}-2\right)>0
$$

for all $n^{\ell}>2$. Indeed, it is simple to check that

$$
\left(n^{\ell}+3\right)\left(n^{\ell}+1\right)\left(-56n^{\ell}-8\left(n^{\ell}\right)^2+\left(n^{\ell}\right)^3-19\right)>0,
$$

for $n^{\ell}\geq 13$. This implies that

$$
6\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}\left(n^{\ell}+3\right)\left(n^{\ell}+1\right)\left(-56n^{\ell}-8\left(n^{\ell}\right)^2+\left(n^{\ell}\right)^3-19\right)>6\frac{n^{\ell}}{n^{\ell}+1}\left(n^{\ell}+3\right)\left(n^{\ell}+1\right)\left(-56n^{\ell}-8\left(n^{\ell}\right)^2+\left(n^{\ell}\right)^3-19\right),
$$

and further

$$
\begin{aligned}
&949n^{\ell}+1313\left(n^{\ell}\right)^2+489\left(n^{\ell}\right)^3+33\left(n^{\ell}\right)^4-6\left(n^{\ell}\right)^5-58+6\sqrt{\frac{n^{\ell}}{n^{\ell}+1}}\left(n^{\ell}+3\right)\left(n^{\ell}+1\right)\left(-56n^{\ell}-8\left(n^{\ell}\right)^2+\left(n^{\ell}\right)^3-19\right)> \\
&\qquad >3\left(n^{\ell}\right)^4+9\left(n^{\ell}\right)^3+191\left(n^{\ell}\right)^2+607n^{\ell}-58,
\end{aligned}
$$

for $n^{\ell}\geq 13$.

This shows that $H\left(\frac{Z}{\gamma};n=n^{\ell}-1,n_D=3\right)>0$ for $n^{\ell}\geq 13$. For $3\leq n^{\ell}\leq 12$ we show that $H\left(\frac{Z}{\gamma};n=n^{\ell}-1,n_D=3\right)>0$ point by point. ■

Lemma C.16 *We show that*

$$\frac{d\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\right)}{dn}=\frac{\partial\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\right)}{\partial\beta}\frac{\partial\beta}{\partial n}>0$$

Proof.

$$\frac{\partial\left(-\frac{1}{2}\beta\left(n\right)\left(\gamma\beta\left(n\right)+2\right)\right)}{\partial\beta\left(n\right)}=-\beta\left(n\right)\gamma-1<0$$

since

$$\begin{aligned}H\left(-\frac{1}{\gamma}\right) &= -2\left(n-1\right)n-\left(\left(2n-1\right)n_D-\left(n-2\right)2n\right)+2\left(n-1\right)n_D\\ &= -2n-n_D<0\end{aligned}$$

$$\beta\left(n\right)>-\frac{1}{\gamma}$$

■

Lemma C.17 *We show that*

$$n_D+2\gamma n^2\frac{\partial\beta\left(n\right)}{\partial n}>0$$

Proof. Using the definition of $\frac{\partial\beta(n)}{\partial n}$

$$n_D\left(-4\left(n-1\right)n\gamma\beta\left(n\right)+\left(\left(2n_I-1\right)n_D-\left(n-2\right)2n\right)\right)-2n^2\left(2\left(-\gamma^2\beta\left(n\right)^2\left(2n-1\right)-\gamma\beta\left(n\right)\left(2\left(n-1\right)-n_D\right)+n_D\right)\right)<0$$

$$4n^2\left(2n-1\right)\gamma^2\beta\left(n\right)^2+\left(-4n^2\left(n_D-2n+2\right)-4nn_D\left(n-1\right)\right)\gamma\beta\left(n\right)+\left(n_D\left(n_D\left(2n-1\right)-2n\left(n-2\right)\right)-4n^2n_D\right)<0$$

Using the definition of $\beta\left(n\right)^2$

$$\left(2n^2+\left(2n-1\right)n_D\right)\frac{\left(n-1\right)n_D+2\beta\left(n\right)\gamma n_I}{n-1}>0$$

since we show below in Lemma $C.19$ that

$$\left(\left(n-1\right)n_D+2\gamma\beta\left(n\right)n\right)>0.$$

■

Lemma C.18 *We show that* $\frac{\beta(n)b(n)}{(b(n)+n\beta(n))^2}$ *is decreasing in* n*.*

Proof.

$$\frac{\beta(n)\,b(n)}{(b(n)+n\beta(n))^2} = \frac{\beta(n)}{(b(n)+n\beta(n))}\frac{b(n)}{(b(n)+n\beta(n))}$$

Then,

$$\begin{aligned}\frac{d}{dn}\left(\frac{\beta(n)\,b(n)}{(b(n)+n\beta(n))^2}\right) &= \frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right)\frac{b(n)}{b(n)+n\beta(n)} \\ &\quad+\frac{\beta(n)}{b(n)+n\beta(n)}\frac{d}{dn}\left(\frac{b(n)}{(b(n)+n\beta(n))}\right)\end{aligned}$$

From Lemma C.20 and Lemma C.19 we know that

$$\frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right)<0 \text{ and } \frac{d}{dn}\left(\frac{b(n)}{(b(n)+n\beta(n))}\right)<0$$

Since $b(n)<0$ and $\beta(n)<0$ the result follows. ■

Lemma C.19 *We show that*

$$\frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right)<0.$$

Proof.

$$\begin{aligned}\frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) &= \frac{\partial}{\partial n}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right)+\frac{\partial}{\partial b(n)}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right)\frac{db(n)}{dn} \\ &\quad+\frac{\partial}{\partial\beta(n)}\left(\frac{\beta(n)}{(b(n)+n\beta(n))}\right)\frac{\partial\beta}{\partial n},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial}{\partial n}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) &= -\frac{\beta(n)^2}{(b(n)+n\beta(n))^2}<0 \\ \frac{\partial}{\partial b(n)}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right) &= -\frac{\beta(n)}{(b(n)+n\beta(n))^2}>0 \\ \frac{\partial}{\partial\beta(n)}\left(\frac{\beta(n)}{(b(n)+n\beta(n))}\right) &= \frac{b(n)}{(b(n)+n\beta(n))^2}<0\end{aligned}$$

Then,

$$\frac{d}{dn}\left(\frac{\beta(n)}{b(n)+n\beta(n)}\right)=\frac{\beta(n)^2}{(b(n)+n\beta(n))^2(n_D-2\gamma\beta(n)n)^2}\left(\left(-2\gamma n^2 n_D\right)\frac{\partial\beta(n)}{\partial n}-\left((n_D-2\gamma\beta(n)\,n)^2+n_D^2\right)\right),$$

where

$$2\gamma n^2 n_D \frac{\partial \beta (n)}{\partial n} + \left((n_D - 2\gamma \beta (n) n)^2 + n_D^2 \right) > 0. \tag{C.8}$$

To see this note that using

$$\frac{\partial \beta (n)}{\partial n} = -\frac{2 \left(-\beta (n)^2 \gamma^2 (2n-1) - \beta\gamma (2(n-1) - N_D) + N_D \right)}{-4(n-1) n\gamma^2 \beta + ((2n-1) N_D - (n-2) 2n) \gamma}$$

Eq.($C.8$) becomes

$$(n_D - 2\gamma\beta(n) n)^2 + 2n^2 n_D \left(-\frac{2\left(-\gamma^2 \beta(n)^2 (2n-1) - \gamma\beta(n)(2(n-1) - n_D) + n_D\right)}{-4(n-1) n\gamma\beta(n) + ((2n-1) n_D - (n-2) 2n)} \right) + n_D^2 > 0$$

The first term is positive. The last two terms can be written as $n_D J(\beta(n))$ where

$$J(\beta(n)) := 2n^2 \left(-\frac{2\left(-\gamma^2 \beta(n)^2 (2n-1) - \gamma\beta(n)(2(n-1) - n_D) + n_D\right)}{-4(n-1) n\gamma\beta(n) + ((2n-1) n_D - (n-2) 2n)} \right) + n_D > 0$$

Rearranging terms

$$J(\beta(n)) = \frac{\left(\left(4\gamma^2 n^2 - 8\gamma^2 n^3\right)\beta^2 + \left(8\gamma n^2 - 8\gamma n^3 + 8\gamma n^2 n_D - 4\gamma n n_D\right)\beta + (1-2n) n_D^2 + \left(6n^2 - 4n\right) n_D\right)}{-(-4(n-1) n\gamma\beta + ((2n-1) n_D - (n-2) 2n))}$$

The denominator is negative. Substituting $\beta(n)^2$, the numerator can be written as

$$-\frac{1}{n-1} ((n-1) n_D + 2\beta(n)\gamma n) \left(2n^2 + (2n-1) n_D\right) < 0$$

since

$$\begin{aligned} ((n-1) n_D + 2\beta(n)\gamma n) &> 0 \\ \gamma\beta(n) &> -\frac{(n-1) n_D}{2n} \end{aligned}$$

because

$$H\left(-\frac{(n-1) n_D}{2n}\right) = -\frac{1}{2} n_I n_D (n_I - 1)(n_D - 2) < 0.$$

Then, $n_D J(\beta(n)) > 0$ and Eq.($C.8$) holds. ■

Lemma C.20 *We show that*

$$\frac{d}{dn}\left(\frac{b\,(n)}{b\,(n)+n\beta\,(n)}\right)<0.$$

Proof. Using the definition of $b\,(n)$ we have

$$\frac{b\,(n)}{b\,(n)+n\beta\,(n)}=\frac{\frac{nn_D\beta(n)}{(n_D-2n\gamma\beta(n))}}{\frac{nn_D\beta(n)}{(n_D-2\gamma n\beta(n))}+n\beta\,(n)}=\frac{n_D}{2\,(n_D-n\gamma\beta\,(n))}$$

Then,

$$\begin{aligned}\frac{d}{dn}\left(\frac{b\,(n)}{b\,(n)+n\beta\,(n)}\right) &= \frac{d}{dn}\left(\frac{n_D}{2\,(n_D-n\gamma\beta\,(n))}\right)=\frac{1}{2}\gamma\frac{n_D}{(n_D-n\gamma\beta\,(n))^2}\left(\frac{d\,(n\beta\,(n))}{dn}\right)\\ &= \frac{1}{2}\gamma\frac{n_D}{(n_D-n\gamma\beta\,(n))^2}\left(n\frac{\partial\beta\,(n)}{\partial n}+\beta\,(n)\right)<0.\end{aligned}$$

■